Friulian Journal of Science **31.** 2021, 65-83

An introduction to one-dimensional inverse eigenvalue problems

ANTONINO MORASSI*

Abstract. Aim of these notes is to present an elementary introduction to classical inverse eigenvalue problems in one dimension. Attention is mainly focused on the Borg's approach to the unique determination of the potential in a Sturm-Liouville differential operator given in canonical form on a finite interval.

Key–words. Inverse eigenvalue problems, Sturm-Liouville operators, Uniqueness, Borg's approach.

1. Introduction. Classical vibration theory is concerned with the determination of the response of a given dynamical system to a prescribed input. These are called *direct problems in vibration* and powerful analytical and numerical methods are available nowadays for their solution. However, when one studies a phenomenon which is governed by the equations of classical dynamics, the application of the model to real life situations often requires the knowledge of constitutive and/or geometrical parameters which in the direct formulation are considered as part of the data, whereas, in practice, they are not completely known or are inaccessible to direct measurements.

Therefore, in several areas of applied science and technology, one has to deal with *inverse problems in vibration*, that is problems in which the roles of the unknowns and the data is reversed, at least in part. For example, one of the basic problems in the direct vibration theory - for infinitesimal undamped free vibrations - is the determination of the natural frequencies and normal modes of the vibrating body, assuming

* Polytechnic Department of Engineering and Architecture, University of Udine, Udine, Italy.
E-mail: antonino.morassi@uniud.it

that the elastic and inertial properties are known. In the context of inverse theory, on the contrary, one is dealing with the construction of a vibrating model that has given (e.g., measured) eigenproperties.

In addition to its applications, the study of inverse problems in vibration has also inherent mathematical interest, since the issues encountered have remarkable features in terms of originality and technical difficulty, when compared with the classical problems of direct vibration theory. In fact, inverse problems do not usually satisfy the Hadamard postulates of well-posedeness, also, in many cases, they are extremely non-linear, even if the direct problem is linear. In most cases, in order to overcome such obstacles, it is impossible to invoke all-purpose, ready made, theoretical procedures. Instead, it is necessary to single out a suitable approach and trade-off with the intrinsic ill-posedeness by using original ideas and a deep use of mathematical methods from various areas. Another specific aspect of the study of inverse problems in vibration concerns the numerical treatment and the development of ad-hoc strategies for the treatment of ill-conditioned, linear and non-linear problems. Finally, when inverse techniques are applied to the study of real problems, additional obstructions arise because of the complexity of mechanical modelling, the inadequacy of the analytical models used for the interpretation of the experiments, measurement errors and incompleteness of the field data. Therefore, of particular relevance for practical applications is to assess the robustness of the algorithms to measurement errors and to the accuracy of the analytical models used to describe the physical phenomenon.

To fix the ideas, let us consider a paradigmatic example of inverse eigenvalue problem taken from structural mechanics. Consider a thin straight rod of length L, made by homogeneous, isotropic linear elastic material with constant Young's modulus E, $E > 0$, and constant volume mass density γ, $\gamma > 0$. The free axial vibration of the rod is governed by the partial differential equation

$$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial w(x,t)}{\partial x}\right) - \gamma A(x)\frac{\partial^2 w(x,t)}{\partial^2 t} = 0, \quad (x,t) \in (0,L) \times (0,\infty), \tag{1.1}$$

where $A = A(x)$ is the area of the cross section and $w = w(x,t)$ is the axial displacement of the cross section of abscissa x evaluated at the time t. The function $A = A(x)$ is assumed to be regular (i.e., $C^2([0,L])$) and strictly positive in $[0, L]$. The end cross-sections of the rod are assumed to be fixed, namely

$$w(0,t) = 0 = w(L,t), \quad t \geq 0. \tag{1.2}$$

If the rod vibrates with frequency ω and spatial shape $X = X(x)$, i.e. $w(x,t) = X(x)\cos(\omega t)$, then the free vibrations are governed by the boundary value problem

$$\begin{cases} (A(x)X'(x))' + \lambda A(x)X(x) = 0, \quad \text{in } (0,L), & (1.3) \\ X(0) = 0 = X(L), & (1.4) \end{cases}$$

where $\lambda = \frac{E}{\gamma}\omega^2$ and $X \in C^2([0,L]) \setminus \{0\}$. The (real) number λ is called eigenvalue of the rod and the corresponding function $X = X(x)$ is the eigenfunction associated to λ. The pair $\{\lambda, X(x)\}$ is a Dirichlet eigenpair. For example, if $A(x) \equiv$ *const.* in $[0,L]$, then $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $X_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$, $n \geq 1$.

We find convenient to rewrite the problem (1.3)-(1.4) in the Sturm-Liouville canonical form by putting

$$y(x) = \sqrt{A(x)}X(x). \tag{1.5}$$

Then, the new eigenpair $\{\lambda, y(x)\}$ solves

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), \quad \text{in } (0,1), & (1.6) \\ y(0) = 0 = y(1), & (1.7) \end{cases}$$

where the potential $q = q(x)$ is defined as

$$q(x) = \frac{(\sqrt{A(x)})''}{\sqrt{A(x)}}, \quad \text{in } (0,1), \tag{1.8}$$

and where, to simplify the notation, we have assumed $L = 1$.

The *direct* eigenvalue problem consists in finding the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and eigenfunctions $\{y_n(x)\}_{n=1}^{\infty}$ of (1.6)-(1.7) for a potential $q = q(x)$ given in $[0,1]$ or, equivalently, for a given cross section $A = A(x)$ of the rod. Conversely, the *inverse* eigenvalue problem consists, for example, in finding information on the potential $q = q(x)$ in $[0,1]$ given the Dirichlet spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of the rod.

In our introductory example we were lead to consider inverse eigenvalue problems for the Sturm-Liouville operator $Ly = -y'' + qy$ on a finite interval. This class of second order, one-dimensional inverse eigenvalue problems can be considered as a rather consolidated topic in the literature of dynamical inverse problems. Fundamental contributions were given in this field by Borg (1946); Levinson (1949); Marchenko (1950); Gel'fand and Levitan (1955); Hochstadt (1973); Hald (1978); McLaughlin (1988), among others, see also the books by Pöschel and Trubowitz (1987), Levitan (1987) and Gladwell (2004) for a comprehensive presentation.

Aim of this note is to present an elementary introduction to the uniqueness problem for these inverse problems following the classical approach by Borg. In the treatment, I plan to give the general ideas of the methods instead of the complete rigorous proofs. These can be tracked down from the original papers. The presentation of the arguments is almost self-contained and prerequisites are basic knowledge of functional analysis (Brezis 1986; Friedman 1982) and complex analysis (Ahlfors 1984; Titchmarsh 1962).

The outline of the paper is as follows. In Chapter 2, some general properties of the direct eigenvalue problem are presented. Borg's approach to uniqueness for the inverse eigenvalue problem is described in Chapter 3.

2. General properties of the direct eigenvalue problem.

2.1 The Dirichlet eigenvalue problem. We begin by recalling some basic properties of the Dirichlet eigenvalue problem (2.1)-(2.2)

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), \quad \text{in } (0,1), & (2.1)\\ y(0) = 0 = y(1), & (2.2) \end{cases}$$

for real-valued square summable potential q on $(0,1)$, i.e. $q \in L^2(0,1)$.

i) There exists a sequence of Dirichlet eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. The eigenvalues are real numbers and $\lim_{n\to\infty} \lambda_n = +\infty$.

ii) The eigenvalues are simple, that is

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ... \tag{2.3}$$

and the eigenspace $\mathcal{U}_n$ associated to the nth eigenvalue λ_n, $n \geq 1$, is given by

$$\mathcal{U}_n = \text{span}\{g_n\}, \tag{2.4}$$

where $g_n = g_n(x)$ is the eigenfunction associated to λ_n satisfying the normalization condition $\int_0^1 g_n^2(x)dx = 1$.

iii) The family $\{g_n\}_{n=1}^{\infty}$ is an orthonormal basis of the space $\mathcal{D}$ of continuous functions which vanish at $x = 0$ and $x = 1$, that is:

$$\int_0^1 g_n(x)g_m(x)dx = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m, \quad n, m \geq 1, \end{cases} \tag{2.5}$$

and for every $f \in \mathcal{D}$ the series

$$\sum_{n=1}^{\infty} c_n g_n(x), \quad \text{with } c_n = \int_0^1 f(x)g_n(x)dx, \tag{2.6}$$

converges uniformly to f in $[0, 1]$.

The above properties can be deduced from abstract spectral theory for self-adjoint compact operators defined on Hilbert spaces, see Brezis (1986) and Friedman (1982). An alternative approach is based on functional theoretical methods and specific properties of the Sturm-Liouville operators, see Titchsmarsh (1962).

2.2 Asymptotic eigenpair estimates. As we will see in the next sections, the asymptotic behavior of eigenpairs plays an important role in inverse spectral theory. With reference to the Dirichlet problem (1.6)-(1.7), let us consider the solution $y = y(x, \lambda)$ to the initial value problem

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), \quad \text{in } (0, 1), & (2.7) \\ y(0) = 0, & (2.8) \\ y'(0) = 1, & (2.9) \end{cases}$$

for some (possibly complex) number λ and for a (possibly complex-valued) L^2 potential q. By considering the right hand side as a forcing term, y can be interpreted as the displacement of an harmonic oscillator with frequency $\sqrt{\lambda}$. Then, by the Duhamel's representation, the function y is the solution of the Volterra linear integral equation

$$y(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x \sin\sqrt{\lambda}(x-t)q(t)y(t)dt, \quad x \in [0,1]. \tag{2.10}$$

It can be shown that there exists a unique solution belonging to $C^1([0,1])$ of (2.10) ($y \in C^2([0,1])$ if q is continuous). Moreover, y is an entire function of order $\frac{1}{2}$ in λ. We recall that a function $f = f(\lambda)$ of the complex variable λ is an analytic function if has the derivative whenever f is defined. If f is analytic in the whole plane, then f is said to be an entire function. Let $M(r) = \max_{|\lambda|=r}|f(\lambda)|$. An entire function $f = f(\lambda)$ has order s if s is the smallest number such that $M(r) \leq \exp(r^{s+\epsilon})$ for any given $\epsilon > 0$, as $r \to \infty$.

By (2.10) one can also deduce that $y(x,\lambda) \approx \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$ for large $|\lambda|$, precisely

$$\left| y(x,\lambda) - \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right| \leq \frac{\|q\|_{L^2}}{|\lambda|}\exp\left(x(\|q\|_{L^2} + |Im\sqrt{\lambda}|)\right), \tag{2.11}$$

uniformly in $[0,1]$, where $\|q\|_{L^2} = \left(\int_0^1 |q(x)|^2 dx\right)^{1/2}$, $\sqrt{\lambda} = Re\sqrt{\lambda} + iIm\sqrt{\lambda}$, $i = \sqrt{-1}$. The zeros of $y(1,\lambda) = 0$ are the eigenvalues $\{\lambda_n\}$ of the Dirichlet problem (1.6)-(1.7) and, in this case, $y = y(x,\lambda_n)$ is the associated eigenfunction.

Estimate (2.11) suggests that higher order zeros of $y(1,\lambda) = 0$ are close to the (square of) zeros of $\sin\sqrt{\lambda} = 0$, that is $\lambda_n \approx (n\pi)^2$ as $n \to \infty$. Precisely, consider the circle $C_n = \{z \in \mathbb{C}|\ |z - n\pi| = \frac{\pi}{4}\}$ for n large enough. One can prove that there exists $N \in \mathbb{N}$ such that for every $n \geq N$ there exists exactly one zero of $y(1,\lambda) = 0$ inside the circle C_n, namely the following eigenvalue asymptotic estimate holds

$$|\sqrt{\lambda_n} - n\pi| < \frac{\pi}{4}. \tag{2.12}$$

The proof of (2.12) is based on a well-known result of complex analysis: the Rouché's Theorem, see (Ahlfors 1984). Let $f = f(z)$, $g = g(z)$ be two analytic functions inside C_n and assume that $|f(z)| < |g(z)|$ on C_n. Then, Rouché's Theorem states that $g(z)$ and $g(z) + f(z)$ have exactly the same number of zeros inside C_n. In order to apply this theorem, let us formally rewrite the function $y(1,\lambda)$ as

$$y(1,\lambda) = \underbrace{\left(y(1,\lambda) - \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} \right)}_{\equiv f(\lambda)} + \underbrace{\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}}_{\equiv g(\lambda)} \tag{2.13}$$

The delicate point consists in showing that $|f(\lambda)| < |g(\lambda)|$ on C_n. Then, recalling that $g(\lambda)$ has exactly one zero ($\sqrt{\widetilde{\lambda}_n} = n\pi$) inside C_n, one obtains (2.12).

Now, inserting the eigenvalue asymptotic estimate (2.12) in estimate (2.11) we get

$$y(x,\lambda_n) = \frac{\sin\sqrt{\lambda_n x}}{\sqrt{\lambda_n}} + O\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty. \tag{2.14}$$

Recalling that $g_n(x) = \frac{y(x,\lambda_n}{\|y(x,\lambda_n)\|_{L^2}}$ we obtain the asymptotic eigenfunction estimate

$$g_n(x) = \sqrt{2}\sin(n\pi x) + O\left(\frac{1}{n}\right), \tag{2.15}$$

which holds uniformly on bounded subsets of $[0,1] \times L^2(0,1)$ as $n \to \infty$. Finally, by iterating the above procedure, the eigenvalue estimate (2.12) can be improved to obtain

$$\lambda_n = (n\pi)^2 + \int_0^1 q(x)dx - \int_0^1 \cos(2n\pi x)q(x)dx + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{2.16}$$

We conclude this section by recalling the complete asymptotic estimates for Dirichlet, Robin and Mixed end conditions. Let α, $\gamma \in \mathbb{R}$ and note that we are assuming $n \geq 0$.

i) Dirichlet end conditions $y(0) = 0 = y(1)$:

$$\lambda_n = ((n+1)\pi)^2 + \int_0^1 q(x)dx - a_{2(n+1)}(q) + O\left(\frac{1}{n}\right), \tag{2.17}$$

$$g_n(x) = \sqrt{2}\sin((n+1)\pi x) + O\left(\frac{1}{n}\right). \tag{2.18}$$

ii) Robin end conditions $\alpha y(0) + y'(0) = 0 = \gamma y(1) + y'(1)$:

$$\lambda_n = ((n)\pi)^2 + 2(\gamma - \alpha) + \int_0^1 q(x)dx + a_{2n}(q) + O\left(\frac{1}{n}\right), \quad (2.19)$$

$$g_n(x) = \sqrt{2}\cos(n\pi x) + O\left(\frac{1}{n}\right). \quad (2.20)$$

iii) Mixed end conditions $y(0) = 0 = \gamma y(1) + y'(1)$:

$$\lambda_n = \left(\left(n + \frac{1}{2}\right)\pi\right)^2 + 2\gamma + \int_0^1 q(x)dx + a_{2n+1}(q) + O\left(\frac{1}{n}\right), \quad (2.21)$$

$$g_n(x) = \sqrt{2}\sin\left(\left(n + \frac{1}{2}\right)\pi x\right) + O\left(\frac{1}{n}\right). \quad (2.22)$$

Here, $a_n \equiv \int_0^1 \cos(n\pi x)q(x)dx$ is the nth Fourier cosine coefficient of q, with $\sum_{n\geq 0} a_n^2 < \infty$.

2.3 Number of zeros of eigenfunctions: the Dirichlet case.

Theorem 2.1:
The nth Dirichlet eigenfunction, $n \geq 1$, has exactly $n-1$ (simple) zeros inside the interval $(0,1)$.

Sketch of the proof for $n = 1$. We follow (Weinberger 1965). By the variational characterization of the lower eigenvalue we know that

$$\mu_1 = F(g_1) = \min_{\varphi \in H_0^1(0,1)\setminus\{0\}} F(\varphi), \quad \text{where } F(\varphi) = \frac{\int_0^1 (\varphi'^2 + q\varphi^2)}{\int_0^1 \varphi^2} \quad (2.23)$$

is the *Rayleigh quotient* of the problem. If g_1 is the first eigenfunction, then also $|g_1|$ is a function associated to the first eigenvalue. In fact, $(|g_1|')^2 = (sgn(g_1)g_1')^2 = (g_1')^2$ and $F(|g_1|) = \mu_1$. Since the geometric multiplicity of every Dirichlet eigenvalue is simple, two eigenfunctions associated to the same eigenvalue must be proportional, namely $|g_1(x)| = cg_1(x)$ in $[0,1]$, where c is a constant with $|c| = 1$. More precisely, if $c = 1$ then $g_1 \geq 0$ in $(0,1)$ and, conversely, if $c = -1$ then $g_1 \leq 0$ in $(0,1)$. In both cases g_1 does not change sign in $[0,1]$. Finally, if $g_1(\overline{x}) = 0$ for some $\overline{x} \in (0,1)$, then $g_1(\overline{x}) = g_1'(\overline{x}) = 0$ and, by the uniqueness of the Cauchy problem for solutions of the Sturm-Liouville operator, $g_1(x) \equiv 0$ in $[0,1]$, a contradiction.

The result for $n \geq 2$ can be obtained by induction and by using the properties of the oscillation character of the solutions, which is defined by the following Sturm Theorems (Titchmarsh 1962).

Theorem 2.2:
Let u, v, be two non trivial, real-valued solutions to

$$u'' + g(x)u = 0, \quad \textit{in } (a,b), \tag{2.24}$$

$$v'' + h(x)v = 0, \quad \textit{in } (a,b), \tag{2.25}$$

where g, $h \in L^2(a,b)$, $-\infty < a < b < \infty$. If $g < h$ a.e. in (a,b) and x_1, x_2 are two consecutive *zeros of u (e.g., $u(x_1) = u(x_2) = 0$, $a \leq x_1 < x_2 \leq b$), then there exists $\overline{x}$, $x_1 < \overline{x} < x_2$, such that $v(\overline{x}) = 0$.*

Theorem 2.3:
Let u, v be solutions to (2.24), (2.25), *respectively, such that*

$$u(a)\cos\alpha + u'(a)\sin\alpha = 0, \quad v(a)\cos\alpha + v'(a)\sin\alpha = 0, \tag{2.26}$$

where $\alpha \in \mathbb{R}$. Let a, b, g, h as above and let m be an integer number, $m \geq 0$. If u has m zeros in $(a,b]$, then v has at least m zeros in $(a,b]$ and the nth zero of v is less than the nth zero of u.

3. Uniqueness: Borg's approach.

3.1 L^2-symmetric potential. Let us consider the Dirichlet eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), & \text{in } (0,1), \qquad (3.1) \\ y(0) = 0 = y(1), & \qquad (3.2) \end{cases}$$

where $q \in L^2(0,1)$ is a real-valued potential. Denote by $\{g_n(x), \lambda_n\}_{n=1}^{\infty}$, $\int_0^1 g_n^2 dx = 1$, the eigenpairs of (3.1)-(3.2). Let us compare the high order eigenvalues of the above problem with those of the *reference* problem with $q \equiv 0$:

$$\begin{cases} z''(x) + \chi z(x) = 0, & \text{in } (0,1), \quad (3.3) \\ z(0) = 0 = z(1), & \quad (3.4) \end{cases}$$

We know that $\chi_n = (n\pi)^2$, $g_n(x,0) = \sqrt{2}\sin(n\pi x)$, $n \geq 1$. Then, by the asymptotic eigenvalue estimate (2.16) we obtain

$$\lambda_n = \chi_n + \int_0^1 q(x)dx - \int_0^1 \cos(2n\pi x)q(x)dx + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{3.5}$$

Generally speaking, it turns out that knowledge of the high order eigenvalues can give information *only* on the average value of q and on the higher order Fourier coefficients of q evaluated on the set $\{\cos(2n\pi x)\}_{n=1}^{\infty}$. Since $\{\sqrt{2}\cos(2n\pi x)\}_{n=1}^{\infty} \cup \{1\}$ is an orthonormal basis of the space of the even functions with respect to $x = \frac{1}{2}$

$$L^2_{even}(0,1) = \{f \in L^2(0,1) |\ f(x) = f(1-x) \text{ a.e. in } (0,1)\}, \tag{3.6}$$

we expect to be able only to extract information from $\{\lambda_n\}_{n=1}^{\infty}$ for the even part of the potential q. These heuristic considerations were made rigorous in the following celebrated theorem by (Borg 1946).

Theorem 3.1:
Let $q \in L^2_{even}(0,1)$. The potential q is uniquely determined by the full Dirichlet spectrum $\{\lambda_n\}_{n=1}^{\infty}$.

The proof is by contradiction. Suppose that there exist another potential $p \in L^2_{even}(0,1)$, $p \neq q$, such that the eigenvalue problem

$$\begin{cases} z''(x) + \lambda z(x) = p(x)z(x), & \text{in } (0,1), \quad (3.7) \\ z(0) = 0 = z(1), & \quad (3.8) \end{cases}$$

has exactly the same eigenvalues of (3.1)-(3.2), i.e. $\lambda_n(p) = \lambda_n(q)$ for every $n \geq 1$. In order to simplify the notation, let us denote by $g_n(q) = g_n(x,q)$, $g_n(p) = g_n(x,p)$ the normalized Dirichlet eigenfunctions associated to λ_n for potential q and p, respectively. Note that the nth Dirichlet eigenfunction is even when n is odd, and is odd when n is even. By multiplying the differential equation satisfied by $g_n(q)$, $g_n(p)$ by $g_n(p)$, $g_n(q)$, respectively, integrating by parts in $(0,1)$ and subtracting, we obtain

$$\int_0^1 (q-p)g_n(p)g_n(q)dx = 0, \quad \text{for every } n \geq 1. \tag{3.9}$$

Borg's proof is subtle and quite involved. Before embarking in the rigorous proof, we provide a rather simple heuristic argument based on it.

By the asymptotic expression of the normalized eigenfunctions $g_n(p)$, $g_n(q)$ of the Dirichlet problem with potentials $p \in L^2_{even}(0,1)$, $q \in L^2_{even}(0,1)$, respectively, we known that $g_n(x,p) = \sqrt{2}\sin(n\pi x)+O(1/n)$, $g_n(x,q) = \sqrt{2}\sin(n\pi x) + O(1/n)$ (here, the first eigenfunction corresponds to $n = 1$). Then, using these asymptotic forms in (3.9) and neglecting $O(1/n)$ terms one finds

$$0 = 2\int_0^1 (q-p)\sin^2(n\pi x)dx = \int_0^1 (q-p)(1-\cos(2n\pi x))dx, \quad n = 1, 2, ... \tag{3.10}$$

Taking the limit as $n \to \infty$, we get $\int_0^1 q = \int_0^1 p$ and then

$$\int_0^1 (q - p)\cos(2n\pi x)dx, \quad n = 0, 1, 2, ..., \tag{3.11}$$

that is all the Fourier coefficients of the even L^2-function $q - p$ vanish, so that $q = p$ almost everywhere in $(0, 1)$.

We return now to the rigorous proof. By the asymptotic eigenvalue estimate (2.16) we have

$$\int_0^1 q dx = \int_0^1 p dx \tag{3.12}$$

and then condition (3.9) can be written as

$$\int_0^1 (q - p)(1 - g_n(p)g_n(q))dx = 0, \quad \text{for every } n \geq 1. \tag{3.13}$$

To find the contradiction it is enough to show that the family $\{1\}\cup\{1-g_n(q)g_n(p)\}_{n=1}^{\infty}$ is a *complete* system of functions in $L^2_{even}(0,1)$. Actually, we shall prove that this family is a basis of $L^2_{even}(0,1)$. We recall that a sequence of vectors $\{v_n\}_{n=1}^{\infty}$ in a separable Hilbert space H is a basis for H if there exists a Hilbert space h of sequences $\alpha = (\alpha_1, \alpha_2, ...)$ such that the correspondence $\alpha \to \sum_{n=1}^{\infty} \alpha_n v_n$ is a linear isomorphism between h and H (that is, an isomorphism which is continuous and has continuous inverse).

Let us introduce the set of functions $\{U_n\}_{n=0}^{\infty}$ defined as

$$\begin{cases} U_0(x) = 1, & (3.14) \\ U_n(x) = \sqrt{2}\left(\int_0^1 g_n(q)g_n(p)dx - g_n(q)g_n(p)\right), \quad n \geq 1. & (3.15) \end{cases}$$

Clearly, $\{U_n\}_{n=0}^{\infty}$ is a bounded subset of $L^2_{even}(0,1)$. We will prove that $\{U_n\}_{n=0}^{\infty}$ is a basis of $L^2_{even}(0,1)$. This immediately implies that also $\{1\} \cup \{g_n(q)g_n(p) - 1\}_{n=1}^{\infty}$ is a basis of $L^2_{even}(0,1)$.

At this point we make use of the following useful result (Pöschel and Trubowitz, 1987, Theorem 3 of Appendix D).

Theorem 3.2:
Let $\{e_n\}_{n\geq 0}$ be an orthonormal basis of an Hilbert space H. Let $\{d_n\}_{n\geq 0}$ be a sequence of elements of H. If

i) $\{d_n\}_{n=0}^{\infty}$ is such that $\sum_{n=0}^{\infty} \|e_n - d_n\|_H^2 < +\infty$,

and

ii) $\{d_n\}_{n=0}^{\infty}$ are linear independent in H,

then $\{d_n\}_{n=0}^{\infty}$ is a basis of H.

We recall that a sequence $\{v_n\}$ in a separable Hilbert space H is linearly independent if $\sum_n c_n v_n = 0$ for some sequence $\{c_n\}$ with $\sum_n c_n^2 < \infty$, then $c_n = 0$ for all n. We apply the above Theorem with $H = L^2_{even}(0,1)$, $\{e_n\}_{n=0}^{\infty} = \{\sqrt{2}\cos(2n\pi x)\}_{n=1}^{\infty} \cup \{e_0 = 1\}$, $d_n = U_n$ for every $n \geq 0$.

Condition i) is easily checked. By the asymptotic eigenfunction estimate (2.15) we have

$$U_n(x) = \sqrt{2}\cos(2n\pi x) + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty \tag{3.16}$$

and therefore

$$\sum_{n\geq 0} \|e_n - U_n\|_{L^2}^2 = \sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right) < \infty. \tag{3.17}$$

The proof of the linear independence stated in condition ii) is more difficult. The original idea of Borg was to find a sequence of bounded functions $\{V_m\}_{m=0}^{\infty} \subset L^2_{even}(0,1)$ such that $\{U_n, V_m\}_{m,n=0}^{\infty}$ is a bi-orthonormal system of functions in $L^2_{even}(0,1)$, that is

$$\begin{cases} (U_n, V_n) = 1, \quad \text{for every } n \geq 0, & (3.18) \\ (U_n, V_m) = 0, \quad \text{for every } m, n \geq 0, \ n \neq m, & (3.19) \end{cases}$$

where $(U_n, V_m) = \int_0^1 U_n(x) V_m(x) dx$. Choose

$$\begin{cases} V_0(x) = 1, & (3.20) \\ V_m(x) = a_m'(x), \quad m \geq 1, & (3.21) \end{cases}$$

with

$$a_m(x) = g_m(x, q) \zeta_m(x, p), \tag{3.22}$$

where $\zeta_m = \zeta_m(x, p)$ is a *suitable* solution of the differential equation

$$\zeta_m'' + \lambda_m \zeta_m = p \zeta_m, \quad \text{in } (0, 1). \tag{3.23}$$

Note that ζ_m is not an eigenfunction of p since ζ_m does not necessarily satisfy the Dirichlet boundary conditions at $x = 0$ and $x = 1$.

By definition, we have

$$(U_0, V_0) = 1, \quad (U_0, V_n) = 0, \quad (U_n, V_0) = 0, \quad n \geq 1. \tag{3.24}$$

Assume $m, n \geq 1$. Since

$$(U_n, V_m) = -\sqrt{2}(g_n(q) g_n(p), a_m') \tag{3.25}$$

and

$$(g_n(q) g_n(p), a_m') = -((g_n(q) g_n(p))', a_m), \tag{3.26}$$

a direct computation shows that

$$\begin{aligned} ((g_n(q) g_n(p))', a_m) &= \frac{1}{2}((g_n(q) g_n(p))', a_m) + \frac{1}{2}((g_n(q) g_n(p))', a_m) = \\ &= \frac{1}{2} \int_0^1 (g_n(q) g_n(p))' g_m(q) \zeta_m(p) - \frac{1}{2} \int_0^1 (g_n(q) g_n(p))(g_m(q) \zeta_m(p))' = \\ &= \frac{1}{2} \int_0^1 g_m(q) \zeta_m(p)(g_n'(q) g_n(p) + g_n(q) g_n'(p)) - \\ &\quad - g_n(q) g_n(p)(g_m'(q) \zeta_m(p) + g_m(q) \zeta_m'(p)) = \\ &= \frac{1}{2} \int_0^1 \left[\zeta_m(p) g_n(p) \det \begin{pmatrix} g_m(q) & g_n(q) \\ g_m'(q) & g_n'(q) \end{pmatrix} + \right. \\ &\quad \left. + g_m(q) g_n(q) \det \begin{pmatrix} \zeta_m(p) & g_n(p) \\ \zeta_m'(p) & g_n'(p) \end{pmatrix} \right] dx. \end{aligned} \tag{3.27}$$

If $m \neq n$, $m, n \geq 1$, then

$$\left(\det \begin{pmatrix} g_m(q) & g_n(q) \\ g'_m(q) & g'_n(q) \end{pmatrix}\right)' = (\lambda_m - \lambda_n) g_m(q) g_n(q) \tag{3.28}$$

and

$$\left(\det \begin{pmatrix} \zeta_m(p) & g_n(p) \\ \zeta'_m(p) & g'_n(p) \end{pmatrix}\right)' = (\lambda_m - \lambda_n) \zeta_m(p) g_n(p). \tag{3.29}$$

Therefore, we have

$$\begin{aligned} &(\lambda_m - \lambda_n)((g_n(q)g_n(p))', a_m) = \\ &\quad = \frac{1}{2} \det \begin{pmatrix} g_m(q) & g_n(q) \\ g'_m(q) & g'_n(q) \end{pmatrix} \det \begin{pmatrix} \zeta_m(p) & g_n(p) \\ \zeta'_m(p) & g'_n(p) \end{pmatrix} \Bigg|_{x=0}^{x=1} = 0, \end{aligned} \tag{3.30}$$

since $g_m(x, q) = g_n(x, q) = 0$ at $x = 0$ and $x = 1$. This means that $(U_n, V_m) = 0$ for $m, n \geq 1$ and $m \neq n$.

Let $m = n$, $m, n \geq 1$. Recalling that

$$\det \begin{pmatrix} \zeta_n(p) & g_n(p) \\ \zeta'_n(p) & g'_n(p) \end{pmatrix} \equiv \text{const}, \quad \text{in } [0, 1], \tag{3.31}$$

the bi-orthonormality condition

$$(U_n, V_n) = \frac{1}{\sqrt{2}} \det \begin{pmatrix} \zeta_n(p) & g_n(p) \\ \zeta'_n(p) & g'_n(p) \end{pmatrix} = 1 \tag{3.32}$$

is satisfied if and only if

$$\zeta_n(\frac{1}{2}, p) g'_n(\frac{1}{2}, p) - \zeta'_n(\frac{1}{2}, p) g_n(\frac{1}{2}, p) = \sqrt{2}. \tag{3.33}$$

The function ζ_n is not uniquely determined by the single condition (3.33). We must impose a second initial condition at $x = \frac{1}{2}$. Recalling that for n odd the function $g_n(p)$ is even and $g'_n(\frac{1}{2}) = 0$, the function $\zeta_n(p)$ can be chosen such that

$$\zeta_n(\frac{1}{2}, p) = 0, \quad \zeta'_n(\frac{1}{2}, p) = -\frac{\sqrt{2}}{g_n(\frac{1}{2}, p)}. \tag{3.34}$$

Then, the function $\zeta_n(p)$ is odd with respect to $x = \frac{1}{2}$. Conversely, if n is even, $g_n(p)$ is odd and the function $\zeta_n(p)$ can be chosen such that

$$\zeta_n(\frac{1}{2}, p) = \frac{\sqrt{2}}{g_n'(\frac{1}{2}, p)}, \quad \zeta_n'(\frac{1}{2}, p) = 0, \tag{3.35}$$

that is $\zeta_n(p)$ is even with respect to $x = \frac{1}{2}$.

In conclusion, for n even, $g_n(p)$ is odd and $\zeta_n(p)$ is even, and then the function $(g_n(p)\zeta_n(p))'$ is even. Similarly, when n is odd, $g_n(p)$ is even and $\zeta_n(p)$ is odd, and then the function $(g_n(p)\zeta_n(p))'$ is still even, and the construction of the family $\{V_n\}_{n=0}^{\infty}$ is complete.

The method presented above can be extended to cover the case of Neumann boundary conditions. Consider the eigenvalue problem

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), & \text{in } (0,1), \qquad (3.36) \\ y'(0) = 0 = y'(1), & \qquad (3.37) \end{cases}$$

with $q \in L^2(0,1)$ real-valued. Denote by $\{\lambda_n, g_n\}_{n=0}^{\infty}$ the eigenpairs. Borg proved the following uniqueness result.

Theorem 3.3 (Borg 1946):
Let $q \in L^2_{even}(0,1)$. The potential q is uniquely determined by the reduced Neumann spectrum $\{\lambda_n\}_{n=1}^{\infty}$.

As in the Dirichlet case, the crucial point is to prove that the family $\{1\} \cup \{g_n(q)g_n(p)\}_{n=1}^{\infty}$ is a complete system of functions of $L^2_{even}(0,1)$.

It is worth noticing that the uniqueness result holds without the knowledge of the lower eigenvalue λ_0. Actually, the lower eigenvalue plays a special role in this inverse problem, as it is shown by the following theorem.

Theorem 3.4 (Borg 1946):
Let $q \in L^2(0,1)$ with $\int_0^1 q = 0$. If the smallest eigenvalue λ_0 of the problem (3.36)-(3.37) *is zero, then $q \equiv 0$ in* $(0,1)$.

In fact, let us denote by y_0 the eigenfunction associated to the smallest eigenvalue λ_0. By oscillatory properties of Neumann eigenfunctions, y_0 does not vanish in $[0,1]$. Then, we can divide the differential equation (3.36) by y_0 and integrate by parts in $(0,1)$:

$$0 = \int_0^1 q = \int_0^1 \frac{y_0''}{y_0} = \int_0^1 \left(\frac{y_0'}{y_0}\right)' + \left(\frac{y_0'}{y_0}\right)^2 = \int_0^1 \left(\frac{y_0'}{y_0}\right)^2. \tag{3.38}$$

By (3.38) we get $y_0' \equiv 0$ in $[0,1]$ and then $q \equiv 0$ in $(0,1)$.

Remark 3.5. It should be noted that the above uniqueness results can not be extended, in general, to the Sturm-Liouville problem with even slightly different boundary conditions, for example

$$\begin{cases} \alpha y(0) + y'(0) = 0, & (3.39) \\ \gamma y(1) + y'(1) = 0, & (3.40) \end{cases}$$

where α, $\gamma \in \mathbb{R}$. Indeed, if $\alpha \neq 0$ and $\gamma \neq 0$, functions analogous to functions U_n, V_n introduced above are not necessarily symmetrical with respect to the mid-point $x = \frac{1}{2}$. (Borg 1946) gave a counterexample in which the eigenvalue problem with the boundary conditions of the type (3.39)-(3.40) does not lead to a complete set of functions in $L^2_{even}(0,1)$.

3.2 Uniqueness: generic L^2 potential. The uniqueness results addressed in the preceding section show that the set of functions $\{g_n(q)g_n(p)\}_{n=0}^{\infty}$, where $g_n(q)$, $g_n(p)$ are the nth eigenfunction corresponding either to Dirichlet or Neumann boundary conditions for potential q and p respectively, are complete in $L^2_{even}(0,1)$, that is in a space of functions which, roughly speaking, has half dimension of the whole space $L^2(0,1)$. To deal with generic $L^2(0,1)$-potentials, the idea by (Borg 1946) was to associate to the original Sturm-Liouville problem another Sturm-Liouville problem such that the set of functions $\{g_n(q)g_n(p)\}_{n=0}^{\infty}$ of the original problem together with the set of functions $\{\overline{g}_n(q)\overline{g}_n(p)\}_{n=0}^{\infty}$ of the associated problem form a complete set of $L^2(0,1)$. In particular, the boundary conditions of the associated problem are chosen so that they produce an asymptotic spectral behavior sufficiently different from that of the initial problem.

Let $q \in L^2(0,1)$ be a real-valued potential. As an example, we shall consider the Sturm-Liouville problem

$$\begin{cases} y''(x) + \lambda y(x) = q(x)y(x), \quad \text{in } (0,1), & (3.41) \\ y(0) = 0, & (3.42) \\ y(1) = 0 & (3.43) \end{cases}$$

and its *associate* eigenvalue problem

$$\begin{cases} \overline{y}''(x) + \lambda \overline{y}(x) = q(x)\overline{y}(x), \quad \text{in } (0,1), & (3.44) \\ \overline{y}(0) = 0, & (3.45) \\ \gamma \overline{y}(1) + \overline{y}'(1) = 0, & (3.46) \end{cases}$$

where $\gamma \in \mathbb{R}$.

Let us denote by $\{\lambda_n(q), y_n(q) = y_n(x, q, \lambda_n(q))\}_{n=0}^{\infty}$ and by $\{\overline{\lambda}_n(q), \overline{y}_n(q) = \overline{y}_n(x, q, \overline{\lambda}_n(q))\}_{n=0}^{\infty}$ the eigenpairs (with normalized eigenfunctions) of the eigenvalue problems (3.41)-(3.43) and (3.44)-(3.46), respectively. Let us introduce the two companions Sturm-Liouville problems related respectively to (3.41)-(3.43) and (3.44)-(3.46) with potential $p \in L^2(0,1)$, namely

$$\begin{cases} z''(x) + \lambda z(x) = p(x)z(x), \quad \text{in } (0,1), & (3.47) \\ z(0) = 0, & (3.48) \\ z(1) = 0 & (3.49) \end{cases}$$

and its *associate* eigenvalue problem

$$\begin{cases} \overline{z}''(x) + \lambda \overline{z}(x) = p(x)\overline{z}(x), \quad \text{in } (0,1), & (3.50) \\ \overline{z}(0) = 0, & (3.51) \\ \gamma \overline{z}(1) + \overline{z}'(1) = 0. & (3.52) \end{cases}$$

Let $\{\lambda_n(p), z_n(p) = z_n(x, p, \lambda_n(p))\}$, $\{\overline{\lambda}_n(p), \overline{z}_n(p) = \overline{z}_n(x, p, \overline{\lambda}_n(p))\}$, $n = 0, 1, \dots$, be the eigenpairs (with normalized eigenfunctions) of the eigenvalue problems (3.47)-(3.49) and (3.50)-(3.52), respectively.

Borg proved the following celebrated uniqueness result by two spectra.

Theorem 3.6 (Borg 1946):
Let q, $p \in L^2(0,1)$. Under the above notation, if $\lambda_n(q) = \lambda_n(p)$ and $\overline{\lambda}_n(p) = \overline{\lambda}_n(q)$ for every $n \geq 0$, then $p = q$.

The proof follows the lines of the corresponding proof for symmetric potentials, but, of course, is more involved. Even in this case, we can present a simple heuristic argument in support of the uniqueness result. For the sake of simplicity let $\gamma = 0$ in (3.46) and in (3.52). Then, by imposing that the Sturm-Liouville problems (3.41)-(3.43), (3.47)-(3.49) and (3.44)-(3.46), (3.50)-(3.52) have the same spectrum $\{\lambda_n\}$, $\{\overline{\lambda}_m\}$, respectively, we have:

$$\int_0^1 (q-p) y_n(q) z_n(p) dx = 0, \quad \int_0^1 (q-p) \overline{y}_m(q) \overline{z}_m(p) dx = 0, \quad (3.53)$$

for $n, m = 0, 1, 2, ...$ Using the asymptotic forms of the eigenfunctions (2.18), (2.22) and neglecting higher order terms, for $n, m = 0, 1, 2, ...$ one finds

$$\int_0^1 (q-p)\sin^2((n+1)\pi x)dx = 0, \quad \int_0^1 (q-p)\sin^2((m+\frac{1}{2})\pi x)dx = 0, \tag{3.54}$$

from which it follows that

$$\int_0^1 (q-p)(1-\cos(2(n+1)\pi x))dx = 0, \int_0^1 (q-p)(1-\cos((2m+1)\pi x)dx = 0, \tag{3.55}$$

for $n, m = 0, 1, 2, ...$, that is

$$\int_0^1 (q-p)\cos(k\pi x)dx = 0, \quad k = 0, 1, 2, ... \tag{3.56}$$

Therefore, $q = p$ a.e. in $(0, 1)$.

4. Conclusions. One of the main issues of the inverse spectral theory for Sturm-Liouville operators in canonical form is the unique determination of the potential. In this note we have presented the classical approach by Borg for symmetric and generic L^2 potential on a finite interval. In a forthcoming note we will present the alternative approach to uniqueness based on function theory arguments initiated by Levinson (1949) and later developed by Hochstadt (1973).

References/ Bibliografie

Ahlfors L.V. (1984). *Complex Analysis*, Third Edition. Singapore: International Student Edition, McGraw-Hill.

Borg G. (1946). Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte. *Acta Mathematica*, 78:1–96.

Brezis H. (1986). *Analisi funzionale: teoria e applicazioni.* Napoli: Liguori Editore.

Friedman A. (1982). *Foundations of Modern Analysis.* Toronto: Dover.

Gel'fand I.M., Levitan B.M. (1955). On the determination of a differential equation from its spectral function. *American Mathematical Society Translations: Series 2*, 1:253–304 (see also *Izvestiya Akademii Nauk Sssr: Seriya Matematicheskaya*, in Russian, (1951), 15:309–360).

Gladwell G.M.L. (2004). *Inverse Problems in Vibration*, Second Edition. Dordrecht: Kluwer Academic Publishers.

Hald O.H. (1978). The inverse Sturm-Liouville problem with symmetric potentials. *Acta Mathematica*, 141:263–291.

Hochstadt H. (1973). The inverse Sturm-Liouville problem. *Communications on Pure and Applied Mathematics*, 26:715–729.

Levinson N. (1949). The inverse Sturm-Liouville problem. *Math. Tidsskr. B.*, 25–30.

Levitan B.M. (1987). *Inverse Sturm-Liouville Problems.* Utrecht: VNU Science Press.

Marchenko V.A. (1950). Concerning the theory of a differential operator of the second order. *Doklady Akademii Nauk SSSR*, 72:457–460.

McLaughlin J.R. (1988). Stability theorems for two inverse spectral problems. *Inverse Problems*, 4:529–540.

Pöschel J., Trubowitz E. (1987). *Inverse Spectral Theory.* New York: Academic Press.

Titchmarsh E.C. (1962). *Eigenfunction Expansion Associated with Second-Order Differential Equations. Part I.* Oxford: Clarendon Press.

Weinberger H.F. (1965). *A First Course on Partial Differential Equations with Complex Variables and Transform Methods.* New York: Dover Publications.