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Recent results on the identification of an open crack in a beam from minimal eigenfrequency data

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Abstract. We review some recent results on the identification of an open crack in a straight elastic beam with smooth variable profile, either under axial or in-plane bending infinitesimal vibration, from the knowledge of a suitable pair of eigenfrequencies. We investigate on sufficient conditions for the unique identification of the crack and we present a constructive algorithm based on the λ-Curves Method. We also discuss a generalization of the methodology to rods with piecewise smooth profile.

Keywords. Damage identification, cracks, resonant and antiresonant frequencies, beams with variable profile, inverse problems.

1. Introduction. The identification of cracks in one-dimensional elements by eigenfrequency data is a topic which has attracted great interest in the scientific community in the last thirty years. We refer, for example, to the papers by Hearn & Testa (1991), Carden & Fanning (2004), Khoo et al. (2004), Humar et al. (2006) for an overview on modal analysis techniques for damage detection in structures.

Most of the research work concerns with the identification of open cracks in uniform beams, see, among other contributions, Adams et al. (1978), Springer et al. (1988), Ruotolo & Surace (1997), Capecchi &

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Vestroni (2000), Cerri & Vestroni (2000), Vestroni & Capecchi (2000), Teughels et al. (2002), Gladwell (2004), Dilena & Morassi (2009), Rubio (2009), Pau et al. (2011), Greco & Pau (2012), and Caddemi & Caliò (2009, 2014) for crack identification by frequency and modal displacement data. Beams with uniform cross-section allow to express the frequency equation in closed-form, and this property makes the inverse problem simpler and more manageable, both from the theoretical and the numerical point of view. In particular, a rather complete theory is now days available for the identification of a single open crack in uniform rods or beams when the damage is *small*, that is when the damaged system can be considered as a perturbation of the undamaged one, see Narkis (1994), Morassi (2001), Dilena & Morassi (2004). Regarding the identification of a finite number of small cracks in rods and beams using natural frequencies, the reader can find an exhaustive analysis of the literature and an original reconstruction algorithm in the interesting article by Shifrin (2016).

The problem of determining a single *not necessarily small* open crack in a longitudinally vibrating uniform rod by frequency data has been recently solved in Rubio et al. (2015a). The crack is modelled by a linearly elastic translational spring located at the damaged cross-section. It was proved in Rubio et al. (2015a) that the knowledge of the first (positive) natural frequency of a free-free rod and the first antiresonant frequency of the driving-point frequency response function of the rod evaluated at one end of the rod uniquely determines the position and the severity of the crack.

Few research works focussed on the identification of not necessarily small cracks in *non-uniform* rods and beams, see, for example, Liang et al. (1992) and Chaudhari & Maiti (2000). In Rubio et al. (2015b), the λ-Curves Method was introduced as useful tool in formulating and solving the inverse problem within this more general context. The λ-Curves Method is mainly based on the study of some refined properties of the eigenfrequencies as functions of the position and severity of the crack, and it leads to a constructive algorithm for solving the diagnostic problem. In this paper we review some recent results we have obtained along this line of research, and we present an extension of the method to axially vibrating rods with piecewise regular profile. The method has been tested on an extended series of numerical simulations, and its stability to errors has been checked both for noisy and experimental data. The interested reader is refereed to the papers Rubio et al. (2015b, 2018, 2020) for more details on theoretical aspects and numerical simulations.

2. Formulation of the inverse problem. Let us consider a longitudinally vibrating free-free thin rod of length L. Denote by $\widehat{A} = \widehat{A}(z)$ the area of the transversal cross-section of the rod, $z \in [0, L]$, and assume that $\widehat{A}$ is a strictly positive, continuously differentiable function in $[0, L]$. The (constant) Young's modulus of the material is denoted by E, $E > 0$; γ is the (constant) volume mass density, $\gamma > 0$. The rod has a single crack at the cross-section of abscissa z_d, with $0 < z_d < L$. The crack is assumed to remain open during vibration and it is modelled as a longitudinal linearly elastic spring with stiffness $\widehat{K}$, see Freund & Herrmann (1976), Adams et al. (1978) and Cabib et al. (2001). The value of $\widehat{K}$ depends on the geometry of the cracked cross-section and on the material properties of the beam. We refer to section 5.2 for a specific expression in the case of rectangular cross-section and transversal crack. The free undamped longitudinal vibration of the rod, with radian frequency ω and spatial amplitude $u = u(x)$, is governed by the following dimensionless eigenvalue problem

$$\begin{cases} (au')' + \lambda a u = 0, \quad x \in (0, s) \cup (s, 1), & (2.1) \\ [[au'(s)]] = 0, & (2.2) \\ K[[u(s)]] = a(s)u'(s), & (2.3) \\ a(0)u'(0) = 0 = a(1)u'(1), & (2.4) \end{cases}$$

where, for a given $x_0 \in [0, 1]$, $x = \frac{z}{L}$, $s = \frac{z_d}{L}$ and

$$A(x) = \widehat{A}(z), \quad a(x) = \frac{A(x)}{A(x_0)}, \quad K = \frac{\widehat{K}L}{EA(x_0)} \in (0, \infty), \quad \lambda = \frac{\gamma L^2 \omega^2}{E}. \tag{2.5}$$

Moreover, we define $[[u(s)]] = (\lim_{x \to s^+} u(x) - \lim_{x \to s^-} u(x))$. Under the above assumptions, there exists a numerable sequence of real, non-negative eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of (2.1)–(2.4) with accumulation point at $+\infty$. The lower eigenvalue $\lambda_0 = 0$ corresponds to a rigid body motion $u(x) = const$, and it is insensitive to damage. We are now in position to state our first result. In particular, to simplify the presentation, we consider a rod with symmetric profile.

Property 1. Let $a = a(x)$ be a strictly positive and continuously differentiable function in $[0,1]$, with $a(x) = a(1-x)$. The measurement of the first two positive natural frequencies of (2.1)–(2.4) allows for the unique determination of the severity K and the location s of the crack, up to the symmetric position $(1-s)$. The identification procedure is constructive.

3. The λ-Curves Method. In order to prove Property 1 via the λ-Curves Method we need some auxiliary results. First, we found convenient to formulate the crack identification problem as an equivalent problem of determining the position and the intensity of a point mass in a rod. The equivalence between the eigenvalue problem for the cracked rod (2.1)–(2.4) and the eigenvalue problem for a longitudinally vibrating rod carrying a point mass at the cracked cross-section is as follows. Let us denote by (λ, u) an eigenpair of (2.1)–(2.4). If $\lambda > 0$, then λ is an eigenvalue of the problem

$$\begin{cases} (bw')' + \lambda bw = 0, \quad x \in (0,s) \cup (s,1), & (3.1) \\ [[w(s)]] = 0, & (3.2) \\ [[bw'(s)]] = -\lambda m w(s), & (3.3) \\ w(0) = 0 = w(1), & (3.4) \end{cases}$$

where

$$w = au' \quad \text{in } (0,s) \cup (s,1), \quad b = a^{-1} \quad \text{in } \in [0,1], \quad m = K^{-1}. \quad (3.5)$$

Conversely, if (λ, w) is an eigenpair of the problem (3.1)–(3.4), then λ, $\lambda > 0$, is an eigenvalue of the problem (2.1)–(2.4) with eigenfunction u such that

$$u = bw' \quad \text{in } (0,s) \cup (s,1), \quad a = b^{-1} \quad \text{in } (0,1), \quad K = m^{-1}. \quad (3.6)$$

In the sequel, basing on the equivalence between the eigenvalue problems (2.1)–(2.4) and (3.1)–(3.4), we reformulate our inverse problem of determining a crack in an axially vibrating (symmetric) free-free rod as the inverse problem of determining the intensity and location of a point mass in an axially vibrating (symmetric) simply supported rod.

In order to study the diagnostic problem we shall often compare the eigenvalues of the problem (3.1)–(3.4) for finite no-vanishing m and for

$s \in (0,1)$, with those obtained by taking $m = 0$ in (3.1)–(3.4). We shall denote by (λ_n^U, w_n^U) the nth eigenpair of the corresponding *unperturbed* (or uncracked) problem. By the variational and Maximum-Minimum formulation, it can be deduced that $\lambda_{n-1}^U \leq \lambda_n \leq \lambda_n^U$, for every $n \geq 1$, where we have defined $\lambda_0^U = 0$.

Our identification algorithm is based on qualitative properties of the functions $\lambda_n = \lambda_n(s, \cdot)$ and $\lambda_n = \lambda_n(\cdot, m)$, that is the λ-m and λ-s curves, respectively, where λ_n is an eigenvalue of (3.1)–(3.4) for a given coefficient b. It can be shown that the study of these properties strongly relies on the explicit expression taken by the first-order eigenvalue derivatives with respect to m and s. More precisely, let $(\lambda, w = w(x))$ be an eigenpair of (3.1)–(3.4). Then, the function $\lambda = \lambda(s, m)$, for $s \in [0,1]$ and $m \in [0, \infty)$, is a continuous function with continuous first order partial derivatives, and we have

$$\frac{\partial \lambda}{\partial s} = -\lambda\, \frac{m w(s)(w'(s^+) + w'(s^-))}{m w^2(s) + \int_0^1 b w^2}, \qquad \frac{\partial \lambda}{\partial m} = -\lambda\, \frac{w^2(s)}{m w^2(s) + \int_0^1 b w^2}, \tag{3.7}$$

where we have defined $w'(s^+) = \lim_{x_0 \to s^+} \left(\frac{dw(x;s,m)}{dx}|_{x=x_0} \right)$ and $w'(s^-) = \lim_{x_0 \to s^-} \left(\frac{dw(x;s,m)}{dx}|_{x=x_0} \right)$.

The dependence of the eigenvalue on the parameter m, for a given position s of the point mass, is analyzed first. The following properties hold:

i) If $w_n^U(s_0) = 0$ for some $s_0 \in [0,1]$, then $\lambda_n(s_0, m) = \lambda_n^U$ for every finite positive m.

ii) If $w_n^U(s_0) \neq 0$ for some $s_0 \in (0,1)$, then $\lambda_n = \lambda_n(s_0, m)$ is a monotonically decreasing function of m in $[0, \infty)$.

iii) If $\lambda_n(s_0, m_0) = \lambda_n^U$ for some $s_0 \in [0,1]$ and $m_0 \in (0, \infty)$, then $w_n^U(s_0) = 0$.

iv) If $w_n(s_0; s_0, m_0) = 0$ for some $s_0 \in [0,1]$ and $m_0 \in (0, \infty)$, then $w_n^U(s_0) = 0$.

A key result concerns the critical points of the λ-s curves. The result is stated here only for the first two eigenvalues of (3.1)–(3.4), since only this set of spectral data will be used to identify the point mass. Let m be given, $0 < m < \infty$. Then $\lambda_1 = \lambda_1(s)$ is a strictly decreasing function in $(0, 1/2)$, and there exists a unique $\widetilde{s} \in (0, 1/2)$ such that $\frac{\partial \lambda_2}{\partial s}(\widetilde{s}) = 0$, that is $\lambda_2 = \lambda_2(s)$ is a strictly decreasing function and a strictly increasing function in $(0, \widetilde{s})$ and in $(\widetilde{s}, 1/2)$, respectively. Recall that $\lambda_i(s) = \lambda_i(1-s)$ for $s \in [0,1]$.

We are now in position to present the identification algorithm. The main steps of the constructive procedure are as follows. Input data $\{\overline{\lambda}_1, \overline{\lambda}_2\}$ are chosen such that $0 < \overline{\lambda}_1 < \lambda_1^U, \quad \lambda_1^U \leq \overline{\lambda}_2 \leq \lambda_2^U$. Note that the upper bound for $\overline{\lambda}_1$ is strict, namely the first eigenvalue is always 'sensitive' to the point mass m. If $\overline{\lambda}_2 = \lambda_2^U$, then the point mass is located at $s = 1/2$ and m can be uniquely determined by solving the equation $\overline{\lambda}_1 = \lambda_1(1/2, m)$. Therefore, in the sequel we shall consider the non-trivial condition $\overline{\lambda}_2 < \lambda_2^U$ and, by symmetry hypothesis, we shall assume $s \in (0, 1/2)$.

We start determining the values m_1^-, m_2^-, $0 < m_i^- < \infty$, $i = 1, 2$, of the parameter m such that $\overline{\lambda}_1 = \lambda_1(1/2, m_1^-)$, $\overline{\lambda}_2 = \lambda_2(s_{2min}, m_2^-)$, where $s_{2min} \in (0, 1/2)$ is the unique point such that $\frac{\partial \lambda_2(s, m_2^-)}{\partial s}|_{s=s_{2min}} = 0$. Note that $m_1^- \neq m_2^-$ and $\max\{m_1^-, m_2^-\} < m$. We distinguish two main cases.

Case 1. If

$$\max\{m_1^-, m_2^-\} = m_1^-, \tag{3.8}$$

then we determine the curve $y = \lambda_2(s, m_1^-)$ in $[0,1]$, see Figure 1 (upper). Let us consider the curves $y = \lambda_2(s, \mathcal{M})$ for $\mathcal{M} > m_1^-$, $\mathcal{M}$ not too large. Let us denote by $P_{2r}(\mathcal{M})$ the intersection point between $y = \lambda_2(s, \mathcal{M})$ and $y = \overline{\lambda}_2$, with the abscissa $s(P_{2r}(\mathcal{M}))$ such that $s(P_{2r}(\mathcal{M})) > s_{2min}$. Moreover, let us denote by $P_1(\mathcal{M})$ the unique intersection point between $y = \lambda_1(s, \mathcal{M})$ and $y = \overline{\lambda}_1$, with $s(P_1(\mathcal{M})) < 1/2$. Then, it can be proved that there exists a unique value of $\mathcal{M}$, say $\widetilde{\mathcal{M}}$, such that $s(P_{2r}(\widetilde{\mathcal{M}})) = s(P_1(\widetilde{\mathcal{M}}))$. The value $\widetilde{\mathcal{M}}$ is the intensity of the mass m and $s = s(P_1(\widetilde{\mathcal{M}}))$ is its position. Case 2. If

$$\max\{m_1^-, m_2^-\} = m_2^-, \tag{3.9}$$

we determine the curve $y = \lambda_1(s, m_2^-)$, denoting by $P_1(m_2^-)$ the unique intersection point between $y = \lambda_1(s, m_2^-)$ and $y = \overline{\lambda}_1$, with abscissa $s_1 = s(P_1(m_2^-)) \in (0, 1/2)$. At this stage, we distinguish two additional subcases. Case 2.- a): Assume that $s_{2min} \leq s_1$. If $s_{2min} = s_1$, then the problem is solved. If $s_{2min} < s_1$, we can repeat the procedure used in Case 1, and the inverse problem has a unique solution, see Figure 1 (lower). Case 2.- b): If $s_{2min} > s_1$, then there exists $m^* > m_2^-$

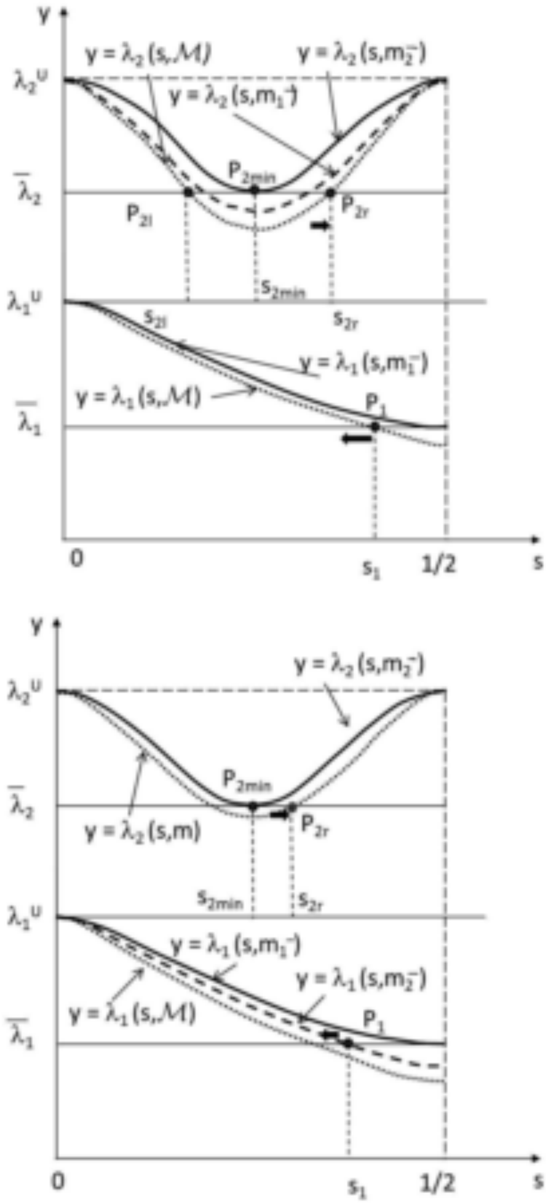

Figure 1: The λ–curves identification algorithm based on first two resonant frequencies: Case 1 (upper) and Case 2 - Subcase a (lower).

such that the intersection point $P_{2l}(m^*)$ between $y = \lambda_2(s, m^*)$ and $y = \overline{\lambda}_2$ satisfies $s(P_{2l}(m^*)) < s(P_1(m_2^-))$, where $P_1(m_2^-)$ is the unique intersection point between $y = \lambda_1(s, m_2^-)$ and $y = \overline{\lambda}_1$. By decreasing the mass value from m^* to m_2^-, there exists a unique value, say $\widetilde{\mathcal{M}}$, such that $s(P_{2l}(\widetilde{\mathcal{M}})) = s(P_1(\widetilde{\mathcal{M}}))$, and the identified parameters are $m = \widetilde{\mathcal{M}}$ and $s = s(P_1(\widetilde{\mathcal{M}}))$.

4. The use of antiresonant eigenfrequency data. Crack at any one of a set of symmetrically placed points of a symmetrical structure produce identical changes to natural frequencies. Therefore, as it was shown in Section 3, the measurement of the first two (positive) natural frequencies of a free-free symmetric rod determines the location of the crack up to a symmetric position. To remove this intrinsic indeterminacy of the inverse problem, appropriate use of antiresonant frequency measurements can be made. Referring to Rubio et al. (2015b) for more details, the following result can be proved.

Property 2. Let $a = a(x)$ be a strictly positive and continuously differentiable function in $[0, 1]$, with $a(x) = a(1 - x)$. The measurement of the first positive natural frequency of (2.1)–(2.4) and of the first antiresonance of the driving point frequency response function of the rod evaluated at one free end allows for the unique determination of the severity K and the location s of the crack. The identification procedure is constructive.

Let $H(\sqrt{\lambda}, x_i, x_0)$ be the frequency response function (FRF) of the axially vibrating rod in (2.1)–(2.4), where x_i, x_o are the abscissa of the excitation point and measurement point, respectively. When $x_i = x_o = 0$, the antiresonances of the FRF $H(\sqrt{\lambda}, 0, 0)$ are the (square root of the) eigenvalues of the rod (2.1)–(2.4) with the homogeneous Neumann end condition $a(0)u'(0) = 0$ replaced by the Dirichlet end condition $u(0) = 0$. Therefore, the main point is the study of the qualitative properties of the λ-m and λ-s curves for the eigenvalue problem for the cracked supported-free rod. Most of the arguments adopted in the proof of Property 1 can be reproduced, with proper modifications, and the identification procedure can be adapted to the present case. We only mention the key fact that, for any given severity of the crack, the first antiresonance $\lambda_{1A} = \lambda_{1A}(s)$ is a strictly increasing function in the interval $(0, 1)$. It is exactly the reduced oscillatory character of λ_{1A} with respect to the second natural frequency of the rod λ_2 that allows the removal of the second solution caused by the symmetry.

Previous results can be generalized to Sturm-Liouville operators more general than those appearing above. Let us consider, for example, the cracked rod under free-free end conditions introduced at the beginning of Section 2. Here, differently from what previously assumed in (2.1)–(2.4), the axial stiffness $a = a(x)$ is not necessarily proportional to the linear mass density $\rho = \rho(x)$ of the rod, and the free undamped vibration is governed by the eigenvalue problem

$$\begin{cases} (au')' + \lambda \rho u = 0, \quad x \in (0,s) \cup (s,1), & (4.1)\\ [[au'(s)]] = 0, & (4.2)\\ K[[u(s)]] = a(s)u'(s), & (4.3)\\ a(0)u'(0) = 0 = a(1)u'(1). & (4.4) \end{cases}$$

The functions $a(x)$ and $\rho(x)$ are assumed to be strictly positive continuously differentiable functions, satisfying the symmetry condition $a(x) = a(1-x)$ and $\rho(x) = \rho(1-x)$ in $[0,1]$. Most of the steps in the previous analysis can be repeated, and it can be shown that Properties 1 and 2 hold also in this case.

5. Extension to rods with piecewise-smooth profile.

5.1 Formulation of the problem and main result. The above results have been proved under the assumption that the rod profile is regular (continuous and continuously differentiable, at least) and symmetric with respect to the mid-point of the rod axis. We will show below that these two a priori assumptions can be removed in proving Property 2.

Let us consider a longitudinally vibrating free-free straight thin rod of length L. The rod is made by linearly elastic material with constant Young's modulus E, $E > 0$, and has uniform volume mass density γ, $\gamma > 0$. Denote by $\widehat{A} = \widehat{A}(z)$ the area of the transversal cross-section of the rod, with $\widehat{A}(z) \geq \widehat{A}_0$ for $z \in [0,L]$, where $\widehat{A}_0 > 0$ is a constant. We shall assume that $\widehat{A} = \widehat{A}(z)$ is *piecewise-C^1-regular* in $[0,L]$, that is, $\widehat{A}$ and its first derivative $\widehat{A}'$ are continuous functions in $[0,L]$ with the exception of the points $\{\widehat{\xi}\}_{i=1}^N$, $0 < \widehat{\xi}_1 < \widehat{\xi}_2 < ... < \widehat{\xi}_N < L$, in which the left and right limit of $\widehat{A}$ and $\widehat{A}'$ there exist and are finite:

$$lim_{z \to \widehat{\xi}_i^\pm} \widehat{A}(z) = \widehat{A}(\widehat{\xi}_i^\pm), \quad lim_{z \to \widehat{\xi}_i^\pm} \widehat{A}'(z) = \widehat{A}'(\widehat{\xi}_i^\pm), \tag{5.1}$$

$i = 1, ..., N$. The rod has a single crack at the cross-section of abscissa z_d, $0 < z_d < L$. The crack is assumed to remain open during vibration and it is modelled as a longitudinally linearly elastic spring with stiffness $\widehat{K}$.

The weak formulation of the eigenvalue problem, in dimensionless form, with radian frequency ω and spatial amplitude $\widehat{u} = \widehat{u}(z)$, is the following: to determine $(\lambda, u = u(x))$, $\lambda \in \mathbb{R}$ and $u \in H^1(0,s) \cup H^1(s,1) \setminus \{0\}$ such that

$$\int_0^1 au'\varphi' dx + K[[u(s)]] \cdot [[\varphi(s)]] = \lambda \int_0^1 au\varphi dx \tag{5.2}$$

for every $\varphi \in H^1(0,s) \cup H^1(s,1)$, where $s = \frac{z_d}{L} \in (0,1)$,

$$K = \frac{\widehat{K}L}{EA(x_0)} \in (0,\infty), \quad \lambda = \frac{\gamma L^2 \omega^2}{E}, \quad \xi_i = \frac{\widehat{\xi}_i}{L}, \quad i = 1, ..., N, \tag{5.3}$$

and $(\cdot)' = \frac{d(\cdot)}{dx}$ means x-differentiation. Let us noticed that, for $0 = \xi_0 < \xi_1 < ... < \xi_i \le s \le \xi_{i+1} < ... < \xi_N < \xi_{N+1} = 1$, the first integral on the left end side of (5.2) is understood as $\int_0^1 (...) = \sum_{j=0}^{i-1} \int_{\xi j}^{\xi j+1} (...) + \int_{\xi i}^{s} (...) + \int_s^{\xi i+1} (...) + \sum_{j=i+1}^{N} \int_{\xi j}^{\xi j+1} (...)$. Under the above assumptions, there exists a numerable sequence of real, non-negative eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of (5.2) such that $0 = \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$ and $\lim_{n\to\infty} \lambda_n = +\infty$.

As before, we introduce the antiresonant frequencies of the driving-point FRF $H(\omega; 0, 0)$ of the cracked rod evaluated at the end cross-section $x = 0$. These antiresonances are the eigenvalues λ_A of the following problem: to determine $(\lambda_A, u_A = u_A(x))$, $\lambda_A \in \mathbb{R}$ and $u_A \in H^1_{(0)}(0,s) \cup H^1(s,1) \setminus \{0\}$ such that

$$\int_0^1 au_A'\varphi' dx + K[[u_A(s)]] \cdot [[\varphi(s)]] = \lambda_A \int_0^1 au_A\varphi dx \tag{5.4}$$

for every $\varphi \in H^1_{(0)}(0,s) \cup H^1(s,1)$, where $H^1_{(0)}(0,s)$ is the set of functions belonging to $H^1(0,s)$ and vanishing at $x = 0$. We shall denote by $0 < \lambda_{1A} < \lambda_{2A} < ... < \lambda_{nA} < ...$ the eigenvalues of (5.4), with $\lim_{n\to\infty} \lambda_{nA} = +\infty$.

We are now in position to state a generalization of Property 2.

Property 3: The measurement of the first (positive) natural frequency λ_1 *of (5.2) and the first antiresonant frequency* λ_{1A} *of (5.4) allows for* *the unique determination of the position s and the severity K of a single open crack in a piecewise-C^1-regular rod. The identification procedure is constructive.*

The proof of Property 3 requires a certain amount of preliminary work, and the derivation of some new mathematical tools respect to the rod with smooth profile. In brief, the proof is based on two main properties. First, as before, the eigenvalue problem for a cracked rod with not smooth profile can be formulated as an equivalent problem for a vibrating rod with a point mass m $(= 1/K)$ at the position s. Second, one can prove that the λ_1-s curve, that is the function $\lambda_1 = \lambda_1(\cdot; m)$ expressing the behavior of the first (positive) eigenfrequency of the free-free cracked rod with respect to the variable s for a given value of m, has exactly one critical point - actually, a minimum - inside the rod axis interval. Moreover, the function $\lambda_{1A} = \lambda_{1A}(s)$ is strictly increasing in $[0, 1)$ and $\frac{\partial \lambda_{1A}}{\partial s}(0) > 0$. It can be shown that these properties follow from the weak formulation of the eigenvalue problem, which, unlike the approach used in Rubio et al. (2015b) for rods with smooth coefficients, allows to easily incorporate the discontinuities of the profile and makes simpler the study of the dependence of the eigenfrequency data on the damage parameters. We refer to Rubio et al. (2020) for details.

5.2 Applications. In order to test the effectiveness of identification to errors on the data, the reconstruction procedure has been applied to pseudo-experimental cases with noisy frequency values. A selected series of results is presented and commented in the sequel.

The specimen is a free-free steel cracked rod with rectangular cross-section of side $B = 0.02$ m (constant) and height $H = H(z)$ given by

$$H(z) = H_0 \left(\frac{1}{4} \left(\frac{z}{L} \right)^2 + 1 \right), \quad 0 \le z < L_s, \tag{5.5}$$

$$H(z) = H_0 \left(\frac{1}{4} \left(\frac{z}{L} \right)^2 + \frac{5}{4} \right), \quad L_s < z \le L, \tag{5.6}$$

where $L = 1$ m, $L_s = 0.6$ m and $H_0 = 0.02$ m. The Young modulus and Poisson ratio of the material are $E = 207$ GPa and $\nu = 0.3$, respectively. The rod has two symmetric transversal cracks, each with front parallel to the side B, of depth $\frac{d}{2} = 4.12$ mm each ($\widehat{K} = 1.4276 \times 10^{10}$ N/m) located at $z_d = 0.35$ m from the left end.

Table 1: Identification of the position z_d, severity $\widehat{K}$ and depth d of a crack in the free-free discontinuous parabolic rod of equation (5.5), by the first (positive) natural frequency and the first antiresonant frequency of $H(\omega; 0, 0)$. Pseudo-experimental case. Level of errors in measured frequencies: $k_1 = 6.0 \times 10^{-4}$ and $k_2 = 9.0 \times 10^{-4}$. Percentage errors: $e_K = 100 \times (\widehat{K}_{est} - \widehat{K})/\widehat{K}$, $e_z = 100 \times (z_{d,est} - z_d)/z_d$, $e_d = 100 \times (d_{est} - d)/d$.

Case		f_1 (Hz)	f_{1A} (Hz)	$z_{d,est}$ (m)	e_z	$\widehat{K}_{est}$ (10^{10} N/m)	e_K	d_{est} (mm)	e_d
Undamaged		2506.10	1151.12						
Damaged	No errors	2493.67	1146.05	0.3496	-0.10	1.4258	-0.12	8.24	0.06
$P1$	$(1+k_1)(f_1)_{exact}$ $(1+k_2)(f_{1A})_{exact}$	2495.16	1147.08	0.3716	6.17	1.7379	21.74	7.49	-8.98
$P2$	$(1-k_1)(f_1)_{exact}$ $(1-k_2)(f_{1A})_{exact}$	2492.17	1145.02	0.3357	-4.08	1.2104	-15.21	8.91	8.22
$P3$	$(1+k_1)(f_1)_{exact}$ $(1-k_2)(f_{1A})_{exact}$	2495.16	1145.02	0.2929	-16.30	1.2817	-10.22	8.61	4.53
$P4$	$(1-k_1)(f_1)_{exact}$ $(1+k_2)(f_{1A})_{exact}$	2492.17	1147.08	0.4296	22.74	1.5571	9.07	8.00	-2.78

The results of damage identification for the perturbed data $(f_1)_{meas} = (1 \pm k_1)(f_1)_{exact}$ and $(f_{1A})_{meas} = (1 \pm k_2)(f_{1A})_{exact}$ are shown in Table 1 ($k_1 = 6 \times 10^{-4}$, $k_2 = 9 \times 10^{-4}$) and in Table 2 ($k_1 = 2.5 \times 10^{-4}$, $k_2 = 4 \times 10^{-4}$). An estimate of the crack depth, obtained from the estimated stiffness $\widehat{K}_{est}$ via equations (5.7) and (5.8), is also provided. Once the stiffness $\widehat{K}$ and the crack location s have been identified, the crack depth can be determined by means of the explicit relationship between the crack depth and the stiffness K. In the present case, denoting by $\frac{d}{2}$ the depth of each side crack, the stiffness $\widehat{K}$ of the elastic spring simulating the damage is expressed as

$$\widehat{K} = \frac{E\widehat{A}(z_d)}{L\delta_l(\nu;\alpha)}, \tag{5.7}$$

where (see Ruotolo & Surace (2004))

$$\begin{aligned}\delta_l(\nu;\alpha) = 2\frac{H(z_d)}{L}(1-\nu^2)(0.7314\alpha^8 - 1.0368\alpha^7 + 0.5803\alpha^6 + \\ + 1.2055\alpha^5 - 1.0368\alpha^4 + 0.2381\alpha^3 + 0.9852\alpha^2)\end{aligned} \tag{5.8}$$

and $\alpha = \frac{d}{H(z_d)}$ is the crack ratio. Therefore, from the identified values of $\widehat{K}$ and s, by knowing $\frac{H(z_d)}{L}$, it is possible, first, to determine $\delta_l(\nu;\alpha)$ and, next, the crack depth d by inverting equation (5.8) with respect to α. For usual values of ν (e.g., $\nu \simeq 0.3$), the function $\delta_l = \delta_l(\nu;\cdot)$ is always uniquely invertible in the interval $\alpha \in [0,1]$, but the interval in which expression (5.8) is accurate is usually smaller.

The analysis of Tables 1–2 shows that the identification procedure is sensitive to errors on the data, although the discrepancies in crack depth estimation are significantly lower than those corresponding to the stiffness predictions, being less than 5 per cent for both the specimens whenever the lower error level is assumed. This different sensitivity is due to the non-linear relationship (5.8) between the stiffness $\widehat{K}$ and the crack depth. However, it should be also noticed that the amount of absolute error that we have considered on the frequency data (e.g., $1-2$ Hz on frequency values ranging from 1200 to 2500 Hz) corresponds to a reasonable magnitude in concrete applications. It is expected that this absolute error could be further reduced by simultaneously improving the experimental procedure and the analytical processing of the data acquired in a vibration test.

Table 2: Identification of the position z_d, severity $\widehat{K}$ and depth d of a crack in the free-free discontinuous parabolic rod of equation (5.5), by the first (positive) natural frequency and the first antiresonant frequency of $H(\omega; 0, 0)$. Pseudo-experimental case. Level of errors in measured frequencies: $k_1 = 2.5 \times 10^{-4}$ and $k_2 = 4.0 \times 10^{-4}$. Percentage errors: $e_K = 100 \times (\widehat{K}_{est} - \widehat{K})/\widehat{K}$, $e_z = 100 \times (z_{d,est} - z_d)/z_d$, $e_d = 100 \times (d_{est} - d)/d$.

Case		f_1 (Hz)	f_{1A} (Hz)	$z_{d,est}$ (m)	e_z	$\widehat{K}_{est}$ (10^{10} N/m)	e_K	d_{est} (mm)	e_d
Undamaged		2506.10	1151.12						
Damaged	No errors	2493.67	1146.05	0.3496	-0.10	1.4258	-0.12	8.24	0.06
$P1$	$(1+k_1)(f_1)_{exact}$ $(1+k_2)(f_{1A})_{exact}$	2494.29	1146.51	0.3587	2.50	1.5472	8.38	7.92	-3.77
$P2$	$(1-k_1)(f_1)_{exact}$ $(1-k_2)(f_{1A})_{exact}$	2493.04	1145.59	0.3423	-2.20	1.3230	-7.33	8.54	3.70
$P3$	$(1+k_1)(f_1)_{exact}$ $(1-k_2)(f_{1A})_{exact}$	2494.29	1145.59	0.3230	-7.71	1.3587	-4.82	8.40	2.02
$P4$	$(1-k_1)(f_1)_{exact}$ $(1+k_2)(f_{1A})_{exact}$	2493.04	1146.51	0.3803	8.65	1.4909	4.44	8.10	-1.61

6. Bending vibrating beams In this section we briefly review the identification of a single open crack in a bending vibrating beam with variable profile via the λ-Curves Method. The interested reader is referred to Rubio et al. (2018) for precise statements of the results and details of the proofs.

We assume that the crack occurs at the cross-section of abscissa z_d, with $0 < z_d < L$, where L is the length of a beam under simply supported end conditions. The crack is assumed to remain open during the vibration and it is modelled as a massless rotational linearly elastic spring with stiffness $\widehat{K}$, see Freund & Herrmann (1976). The free undamped bending vibration of the beam with radian frequency ω and spatial amplitude $u = u(x)$ ($x = z/L$, $z \in [0, L]$) is governed by the following eigenvalue problem (written in dimensionless form)

$$\begin{cases} (ju'')'' - \lambda\rho u = 0, \quad \text{in } (0,s) \cup (s,1), & (6.1) \\ u(x) = u''(x) = 0, \quad \text{at } x = 0 \text{ and } x = 1, & (6.2) \\ [[u(s)]] = [[(ju'')(s)]] = [[(ju'')'(s)]] = 0, & (6.3) \\ K[[u'(s)]] = j(s)u''(s), & (6.4) \end{cases}$$

where $s = z_d/L$, $s \in (0,1)$, $K = \widehat{K}L/EI_0$, $K \in (0,\infty)$, $\lambda = L^4\omega^2/EI_0$ and $I_0 = CL^4$, with $C > 0$ a suitable absolute constant. Here, $\rho = \rho(x)$ is the mass density per unit length, and $j = j(x)$ is the (dimensionless) second moment of area about the axis through the centroid of the cross-section, at right angles to the plane of vibration (the neutral axis). We assume that $j(x)$ and $\rho(x)$ are positive regular functions in $[0, 1]$, and they are symmetric with respect to the mid-point of the beam axis.

The study of the inverse problem follows the lines of the corresponding analysis for the axial vibration problem, albeit with significant differences. The analysis is based on three main steps. First, it is shown that the eigenvalue problem for the cracked beam (6.1)–(6.4) can be transformed in an equivalent eigenvalue problem for a simply-supported beam carrying a point mass $m = 1/K$ at the cracked cross-section s, with suitable bending stiffness and mass density coefficients. Therefore, as in the axial vibration case, the crack detection problem is transformed into the equivalent problem of determining the location s and magnitude m of a point mass from a suitable pair of natural frequencies. In the second step, we study the $\lambda - m$ and $\lambda - s$ curves. The analysis is still based on the explicit determination of the eigenvalue derivatives with respect to the parameters s and m and on specific properties of the eigenpairs of the cracked beam. Under a technical a-priori assumption on the zeros of a suitable function determined in terms of the eigenfunctions of the problem, the above properties are used in the third and last step to define the constructive algorithm of the λ-Curves Method for solving the inverse problem. More precisely, it is shown that the analogue of Property 1 of the axial case is true, that is, the crack can be uniquely determined, up to a symmetric position, from the knowledge of the first two natural frequencies of the beam.

For the sake of completeness, we conclude this section by making some remarks on the technical differences we have found in the present analysis and in the study of the analogous inverse problem of detecting a single open crack in a longitudinally vibrating rod with variable profile considered in Section 3. A first hindrance is connected with the study of qualitative properties of the eigenfunctions of the cracked beam, such as, the number of zeros and interlacing properties between the zeros of eigenfunctions and their derivatives. This study can be carried out in the axial context by extending classical Sturm-Liouville techniques for the undamaged rod to a rod with a crack. Sturm-Liouville methods are not easily extendable to fourth order operators and, therefore, we were forced to follow a different approach, mainly based on the study of the oscillatory character of the statical Green's function of the cracked beam. A second obstruction is connected with the study of the qualitative behavior of the λ–s curves. It can be shown that the argument used in the second order case does not apply to the fourth-order case. The technique we have adopted here is different and it is essentially based on a *deformation argument* which allowed us to reduce the analysis to the study of the zeros of a suitable function defined on the undamaged configuration. It is precisely at this point that, in order to apply the deformation argument, we have introduced a Vanishing Condition, that is an *a priori* assumption on the zeros of a suitable function determined in terms of the eigenfunctions of the cracked beam. It can be shown that this assumption is actually a property of the problem in the case of small damage (Rubio et al. (2018)). In addition, by means of a different approach, it was shown that the Vanishing Condition can be omitted when the simply-supported beam is *uniform*, without introducing any restriction on the damage severity (Rubio et al. (2016)). Whether the Vanishing Condition may or may not be definitively removed from the analysis of the inverse problem remains an open question, at the moment.

7. Conclusions This paper has been devoted to review some recent results on the inverse problem of identifying a single open crack in a vibrating beam by minimal natural frequency and antiresonant frequency data. The beam is considered with variable profile and the crack severity is not necessarily small. Sufficient conditions for the unique identification of the crack location and severity in terms of the frequency data have been established for beams under axial or bending vibration. The analysis leads to a constructive damage identification algorithm, called λ-Curves Method.

Among possible extensions of the present results, we mention the open problem of identifying a single crack in multi-span beams or in one-story frames, and the determination of multiple cracks. However, it should be noted that some of the mathematical tools we have adopted in the analysis of a single beam with a single crack may have no straightforward generalization to those cases, and it is likely that new ideas are needed to deal with these challenging inverse problems.

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