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A mechanical model for small vibrations of an elliptic spider web

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Abstract. A continuous membrane model for the innitesimal deformation of a spider web was recently proposed by Morassi, Soer and Zaera (2017) in the context of axially-symmetric webs. In this note we derive an extension of the model to spider webs with elliptic shape. The analysis of the tensile pre-stress acting on the referential conguration and the free transverse vibration of a supported elliptical web are studied in detail.

1. Introduction. This paper continues a line of research initiated in Morassi et al. (2017) and aimed at developing a mechanical model for spider webs. The spider web is a complex biological-mechanical system that has attracted great interest in the scientific literature of the last four decades, both from the biological and biomechanical point of view. We refer to the introductory section in Morassi et al. (2017) and in Mortimer et al. (2016) for an updated overview of the state-of-the-art and for a discussion on the usefulness of modelling-based approaches to the study of the dynamical response of spider webs. Here, we recall that the first two-dimensional discrete model of spider web was proposed by Aoyanagi & Okumura (2010, 2015). The model was formed by a finite number of radial and circumferential threads, and each thread was described as a stretched spring subject to pre-stress tensile force in

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the referential configuration. The model was used to determine the pre-stress state in an intact axially-symmetric web, and in a web damaged by removing some circumferential threads. The discrete model by Aoyanagi and Okumura was purely static, and its possible use for the study of either in-plane or out-of-plane response was not investigated, not even under the hypothesis of small deformations of the web. In Morassi et al. (2017) a continuum membrane model for the infinitesimal deformation of a spider orb-web was proposed. The model was derived for a specific class of spider orb-webs, namely, the axially-symmetric webs. The actual discrete web, formed by a finite number of radial and circumferential threads, was approximated by a continuous elastic membrane on the assumption that the spacing between threads is small enough. The continuous membrane has a specific fibrous structure which is inherited from the original discrete web, and it is subject to tensile pre-stress in the referential configuration. The out-of-plane static equilibrium and the free transverse and in-plane vibration of a supported circular orb-web were studied in detail, together with the description of the tensile pre-stress acting in the referential configuration.

Although the model proposed by Morassi, Soer and Zaera can be adapted to reproduce general geometries, in Morassi et al. (2017) the attention was restricted to circular-shaped webs in which the circumferential threads belong to concentric circles. The main goal of this note is to extend the analysis developed in Morassi et al. (2017) to spider webs having elliptic shape, that is webs in which the fibrous structure of the continuous membrane is formed by straight threads and elliptical threads.

The analysis of the elliptic geometry turns out to be far from being trivial and, in fact, it requires the introduction of additional a priori hypotheses - with respect to the circular case - to allow a reasonable analytical treatment of the problem. Among these assumptions, one should recall the hypotheses chosen for the determination of the initial pre-traction state in the spider web, both with the *auxiliary* and the *catching* spiral. Another difference from the circular case is the impossibility of separating the radial variable from the angular one in the study of transverse vibration modes, which instead was feasible in case of circular symmetry. Concerning this last point, it is shown that the natural frequencies can be estimated from above and below in terms of the corresponding ones of the circular web, and that the approximation is as good as the elliptic shape is close to a disc.

2. Kinematics. The elliptic spider web considered in this paper is a net formed by two families of intersecting threads, see Figure 1. In a referential configuration $\mathcal{B}_k$, one family coincides with the radial directions passing through the origin O of a two-dimensional Cartesian coordinate system $\{O, X_1, X_2\}$ (*radial threads*), and the other family is formed by homotopic ellipses (*elliptic threads*) having diameters along the axes X_1, X_2 of length $2aR$, $2bR$, respectively, where a, $b \in \mathbb{R}$ and $R > 0$ is a given length. The threads of each family are assumed to be close enough to

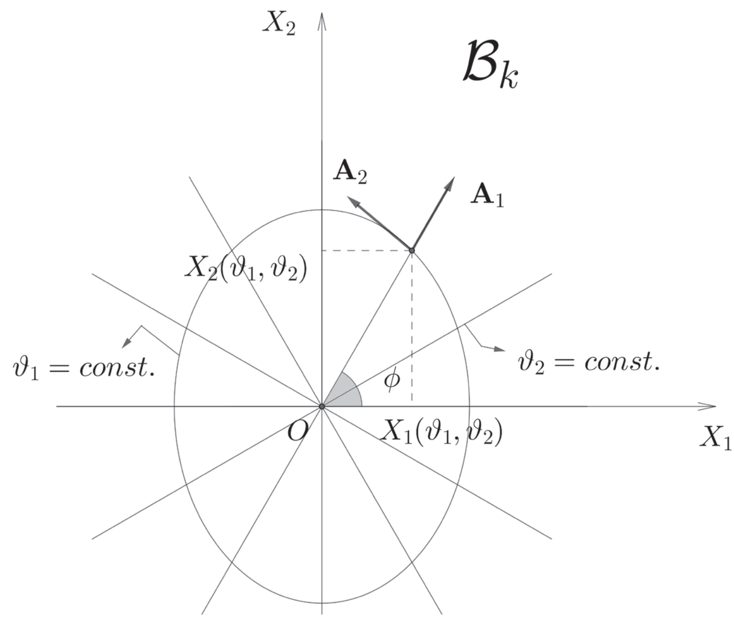


Figure 1: Referential configuration, parametric representation and covariant basis.

each other, so that the web can be described as a two-dimensional elliptic continuous membrane. More precisely, the referential placement $\mathbf{X}$ of the particle X in $\boldsymbol{B}_k$ is given by

$$\mathbf{X} = X_1(\vartheta_1, \vartheta_2)\mathbf{E}_1 + X_2(\vartheta_1, \vartheta_2)\mathbf{E}_2 = \vartheta_1(a\cos\vartheta_2\mathbf{E}_1 + b\sin\vartheta_2\mathbf{E}_2), \quad (2.1)$$

$$\vartheta_1 = \rho,\ \vartheta_2 = \phi,\ \vartheta_1 \in [0, R],\ \vartheta_2 \in [0, 2\pi], \tag{2.2}$$

where $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3 = \mathbf{E}_1 \mathrm{x}\, \mathbf{E}_2\}$ is the canonical basis of $\mathbb{R}^3$, that is $\mathbf{E}_i \cdot \mathbf{E}_j = \delta_{ij}$, with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, $i, j = 1, 2, 3$. Here, "$\times$" and "$\cdot$" denote the vector and scalar product in $\mathbb{R}^3$, respectively. With reference to Figure 1, the radial threads in $\mathcal{B}_k$ coincide with the coordinate curves $\vartheta_2 =$ constant, whereas the elliptical threads are formed by the points with Cartesian coordinates (X_1, X_2) satisfying the equation

$$\frac{X_1^2}{a^2} + \frac{X_2^2}{b^2} = \rho^2, \qquad \rho \in [0, R]. \tag{2.3}$$

Our analysis is developed under the condition

$$b > a \tag{2.4}$$

and, in particular, we shall consider the realistic range of values

$$\frac{b}{a} \in [1.1, 1.3], \tag{2.5}$$

which is roughly satisfied in most of the real spider webs, see, for example, Figure 2. Note that the axially-symmetric geometry of the web, which was consider in Morassi et al. (2017), is obtained assuming $a = b = 1$.
The unit tangent vector to the threads of the αth family is $\frac{\mathbf{A}_\alpha}{|\mathbf{A}_\alpha|}$, where $\mathbf{A}_\alpha = \frac{\partial \mathbf{X}}{\partial \vartheta_\alpha} \equiv \mathbf{X}_{,\alpha}$, $\alpha = 1, 2$, are given by

$$\mathbf{A}_1 = a \cos\vartheta_2 \mathbf{E}_1 + b \sin\vartheta_2 \mathbf{E}_2, \tag{2.6}$$

$$\mathbf{A}_2 = \vartheta_1(-a \sin\vartheta_2 \mathbf{E}_1 + b \cos\vartheta_2 \mathbf{E}_2), \tag{2.7}$$

and $|\mathbf{A}_\alpha| = (\mathbf{A}_\alpha \cdot \mathbf{A}_\alpha)^{\frac{1}{2}}$. Here, $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3 = \mathbf{E}_3\}$ is the covariant basis at a point $\mathbf{X} \in \mathcal{B}_k$, and $\{\mathbf{A}^1, \mathbf{A}^2, \mathbf{A}^3 = \mathbf{A}_3\}$ is the contravariant basis at the same point, with $\mathbf{A}^\alpha \cdot \mathbf{A}_\beta = \delta^\alpha_\beta$, where $\delta^\alpha_\beta = 1$ if $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ if $\alpha \neq \beta$, $\alpha, \beta = 1, 2$, namely,

$$\mathbf{A}^1 = \frac{\cos\vartheta_2}{a}\mathbf{E}_1 + \frac{\sin\vartheta_2}{b}\mathbf{E}_2, \tag{2.8}$$

$$\mathbf{A}^2 = \frac{1}{\vartheta_1}\left(-\frac{\sin\vartheta_2}{a}\mathbf{E}_1 + \frac{\cos\vartheta_2}{b}\mathbf{E}_2\right). \tag{2.9}$$

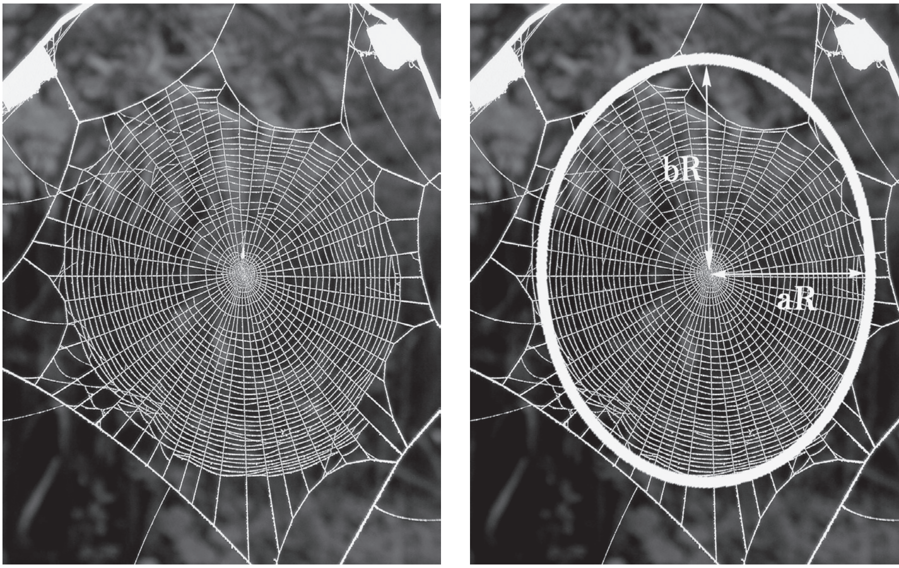


Figure 2: Real spider-web geometry (left) and its elliptical approximation (right).

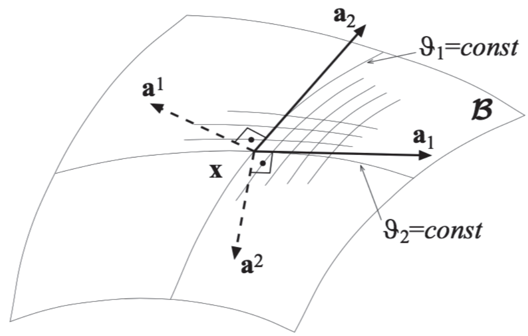
Our analysis is restricted to infinitesimal deformation assigned on the referential configuration $\mathcal{B}_k$. The actual placement of a particle $\mathbf{X} \in \mathcal{B}_k$ at a given time t (which is omitted here, to simplify the notation) is denoted by $\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X})$. The smooth displacement vectorial field $\mathbf{u} : \mathcal{B}_k \to \mathbb{R}^3$ is represented as

$$\mathbf{u} = \sum_{\alpha=1}^{2} u^\alpha \mathbf{A}_\alpha + u^3 \mathbf{A}_3, \tag{2.10}$$

where u^α, $\alpha = 1, 2$, are the contravariant components of $\mathbf{u}$. Note that, hereinafter, Greek indices assume values 1, 2, and summation of the index is explicitly indicated. The assumption of infinitesimal deformation requires that

$$\max\left(\frac{|\mathbf{u}(\mathbf{X})|}{\operatorname{diam}(\mathcal{B}_k)} + \frac{\partial \mathbf{u}(\mathbf{X})}{\partial \mathbf{X}}\right) < \varepsilon, \qquad \mathbf{X} \in \mathcal{B}_k, \tag{2.11}$$

where $\varepsilon \in (0, 1)$ is a given number, and where all the quantities of order $O(\varepsilon^\tau)$, with $\tau > 1$, are neglected. Finally, we denote by $\frac{\mathbf{a}_\alpha}{|\mathbf{a}_\alpha|}$



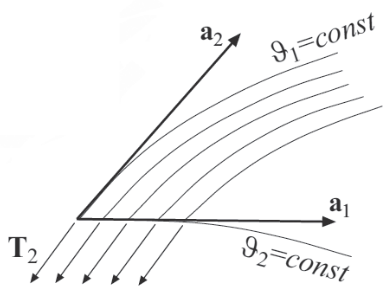


Figure 3: Actual configuration, covariant and contravariant basis, and internal force assumption.

the unit tangent vector to the threads of the αth family in the actual configuration $\mathcal{B}$ of the membrane, see Figure 3, namely

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{x}}{\partial \vartheta_\alpha} \equiv \mathbf{x}_{,\alpha} = \mathbf{A}_\alpha + \mathbf{u}_{,\alpha}, \qquad \alpha = 1, 2, \tag{2.12}$$

where

$$\mathbf{a}_1 = (1 + u^1_{,1})\mathbf{A}_1 + \left(u^2_{,1} + \frac{u^2}{\rho} \right) \mathbf{A}_2 + u^3_{,1}\mathbf{E}_3, \tag{2.13}$$

$$\mathbf{a}_2 = (u^1_{,2} - \rho u^2)\mathbf{A}_1 + \left(1 + u^2_{,2} + \frac{u^1}{\rho} \right) \mathbf{A}_2 + u^3_{,2}\mathbf{E}_3. \tag{2.14}$$

The contravariant basis $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$ in a point $\mathbf{x}$ of $\mathcal{B}$ is defined as $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$, $\mathbf{a}^3 = \mathbf{a}_3 = \frac{\mathbf{a}_1 \mathrm{x}\, \mathbf{a}_2}{|\mathbf{a}_1 \mathrm{x}\, \mathbf{a}_2|}$, and we have

$$\begin{aligned}\mathbf{a}^1 = {} & \frac{1}{\rho^2 a^2 b^2}\left[(A_{12}(u^1_{,2} - \rho u^2) + A_{22}(1 - u^1_{,1}))\right]\mathbf{A}_1 + \\ & + \frac{1}{\rho^2 a^2 b^2}\left[(-A_{11}(u^1_{,2} - \rho u^2) - A_{12}(1 - u^1_{,1}))\right]\mathbf{A}_2 + \\ & + \frac{1}{\rho^2 a^2 b^2}\left[(A_{22}u^3_{,1} - A_{12}u^3_{,2})\right]\mathbf{E}_3,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\mathbf{a}^2 = {} & \frac{1}{\rho^2 a^2 b^2}\left[\left(-A_{22}\left(u^2_{,1} + \frac{u^2}{\rho}\right) - A_{12}\left(1 - u^2_{,2} - \frac{u^1}{\rho}\right)\right)\right]\mathbf{A}_1 + \\ & + \frac{1}{\rho^2 a^2 b^2}\left[\left(A_{12}\left(u^2_{,1} + \frac{u^2}{\rho}\right) + A_{11}\left(1 - u^2_{,2} - \frac{u^1}{\rho}\right)\right)\right]\mathbf{A}_2 + \\ & + \frac{1}{\rho^2 a^2 b^2}\left[(A_{11}u^3_{,2} - A_{12}u^3_{,1})\right]\mathbf{E}_3.\end{aligned} \tag{2.16}$$

3. Fiber densities. We assume that the radial threads in $\mathcal{B}_k$ are equally spaced in the plane angle 2π, and we also assume that the elliptical threads are equally spaced along the radial direction. Therefore, denoting by $\bar{d}_1, \bar{d}_2$ the thread densities in $\mathcal{B}_k$ of the radial and elliptical threads, we have

$$\bar{d}_1 = \frac{C^\rho}{\rho\sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}}, \tag{3.1}$$

$$\bar{d}_2 = \frac{C^\phi}{\sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi}}, \tag{3.2}$$

where the two positive constants C^ρ, C^ϕ are the number $\overline{\#}_\rho$ of radial threads per unit plane angle and the number $\overline{\#}_\phi$ of elliptical threads per unit length along the radial direction in $\mathcal{B}_k$, respectively. With reference to Figure 4, the expression (3.1) of $\bar{d}_1$ guarantees that the number of radial threads crossing the two elliptic arcs A_1B_1 (corresponding to $\rho = \rho_1$) and A_2B_2 ($\rho = \rho_2 > \rho_1$) coincide. We have

$$\overline{\#}_\rho(A_1B_1) = \bar{d}_1(A_1B_1)ds_1, \qquad \overline{\#}_\rho(A_2B_2) = \bar{d}_1(A_2B_2)ds_2, \tag{3.3}$$

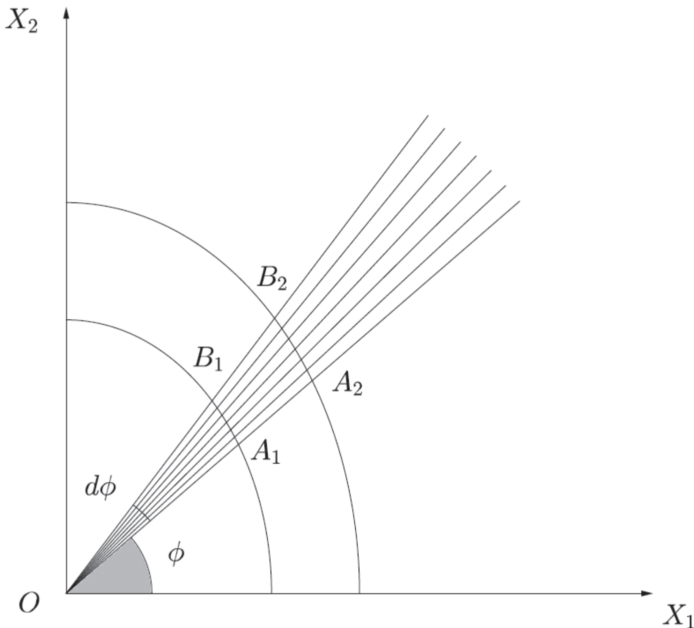


Figure 4: Two generic elliptical threads intercept the same number of radial threads.

where ds_1, ds_2 is the length of the arcs A_1B_1, A_2B_2, respectively. Assuming the points A_α, B_α given by $A_\alpha = (\rho_\alpha, \phi)$, $B_\alpha = (\rho_\alpha, \phi + d\phi)$, $\alpha = 1, 2$, where ϕ is a given angle and $d\phi$ is an infinitesimal increment, by (2.1) we obtain $ds_\alpha = \rho_\alpha \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}\, d\phi$, $\alpha = 1, 2$. Then, using (3.1) in (3.3), we obtain $\overline{\#}_\rho(A_1B_1) = \overline{\#}_\rho(A_2B_2)$. Similarly, the expression (3.2) of $\bar{d}_2$ allows the conservation of the number of elliptical threads crossing the segments $\overline{A_1B_1}$ (corresponding to $\phi = \phi_1$) and $\overline{A_2B_2}$ ($\phi = \phi_2 > \phi_1$), where $A_\alpha = (\rho, \phi_\alpha)$, $B_\alpha = (\rho + d\rho, \phi_\alpha)$, $\alpha = 1, 2$, and $d\rho$ is an infinitesimal increment of the parameter ρ. With reference to the Figure 5 we have

$$\overline{\#}_\phi(A_1B_1) = \bar{d}_2(A_1B_1)ds_1, \quad \overline{\#}_\phi(A_2B_2) = \bar{d}_2(A_2B_2)ds_2, \tag{3.4}$$

where ds_1, ds_2 is the length of the straight segments $\overline{A_1B_1}$, $\overline{A_2B_2}$, respectively. By (2.1) we obtain $ds_\alpha = \sqrt{a^2 \cos^2 \phi_\alpha + b^2 \sin^2 \phi_\alpha}\, d\rho$, $\alpha = 1, 2$, and, therefore, by (3.2), we obtain $\overline{\#}_\phi(A_1B_1) = \overline{\#}_\phi(A_2B_2)$.

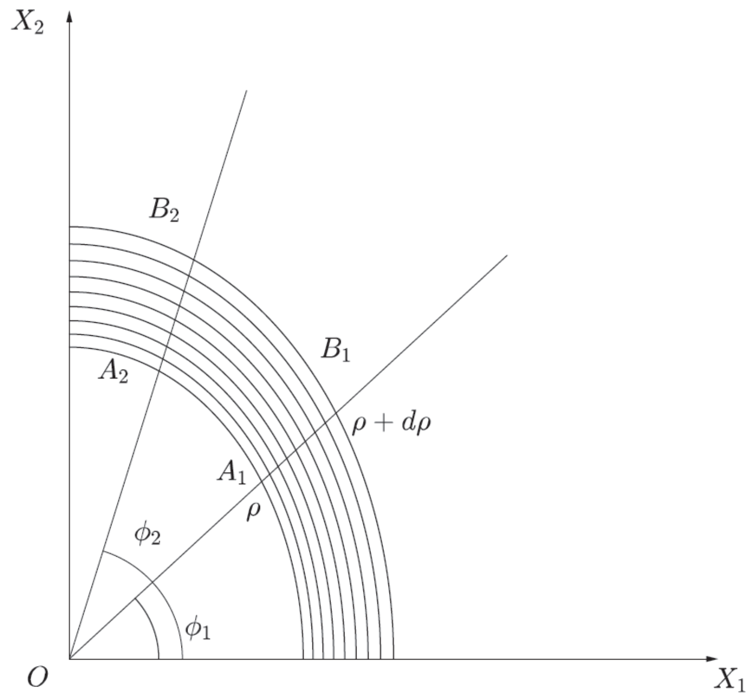


Figure 5: Two generic radial threads intercept the same number of elliptical threads.

It should be noticed that the present analysis of the fiber densities can be extended also to include more general situations in which, for example, $C^\rho = C^\rho(\phi)$ and $C^\phi = C^\phi(\rho)$. Hereinafter, for the sake of simplicity, uniform fiber densities were assumed.

It is understood in our deformation analysis that no slippage can occur between fibers belonging either to the same family or to two different families of threads, so that each given particle has exactly the same two fibers passing through it at each stage of the deformation process. Under this assumption, the expression of the fiber densities d_1, d_2 in the actual configuration $\mathcal{B}$ can be obtained by postulating the conservation of the number of threads crossing a material fiber lying on a coordinate curve in $\mathcal{B}_k$ and the corresponding crossing its image after the deformation. It follows that

$$d_1 = \bar{d}_1 \sqrt{\left(\frac{A_{22}}{a_{22}}\right)}, \quad d_2 = \bar{d}_2 \sqrt{\left(\frac{A_{11}}{a_{11}}\right)}, \tag{3.5}$$

where $A_{\alpha\alpha} = \mathbf{A}_\alpha \cdot \mathbf{A}_\alpha$, $a_{\alpha\alpha} = \mathbf{a}_\alpha \cdot \mathbf{a}_\alpha$, $\alpha = 1, 2$. After linearization, we obtain

$$d_1 = \bar{d}_1 \left(1 - u^2_{,2} - \frac{u^1}{\rho} - r_1 \left(\frac{u^1_{,2}}{\rho} - u^2\right)\right), \tag{3.6}$$

$$d_2 = \bar{d}_2 \left(1 - u^1_{,1} - r_2(\rho u^2_{,1} + u^2)\right), \tag{3.7}$$

with the numbers r_1, r_2 defined as

$$r_1 = \frac{(b^2 - a^2)\sin 2\phi}{2(a^2 \sin^2\phi + b^2 \cos^2\phi)}, \tag{3.8}$$

$$r_2 = \frac{(b^2 - a^2)\sin 2\phi}{2(a^2 \cos^2\phi + b^2 \sin^2\phi)}. \tag{3.9}$$

4. Internal contact forces and equilibrium equations. The analysis follows the arguments shown in Morassi et al. (2017). For reader's convenience, the essential aspects are recalled in the sequel.

We assume that the internal force on an arc element of section along the αth family of threads in the actual configuration $\mathcal{B}$ is a tensile force parallel to the αth coordinate curve, i.e., parallel to $\frac{\mathbf{a}_\alpha}{|\mathbf{a}_\alpha|}$, and we denote by $\mathbf{n}\left(\mathbf{x}, \frac{\mathbf{a}_\alpha}{|\mathbf{a}_\alpha|}\right)$ the force per unit length acting on an arc of the actual surface $\mathcal{B}$ having unit normal $\frac{\mathbf{a}^\alpha}{|\mathbf{a}^\alpha|}$, $\alpha = 1, 2$. The external force field acting on the deformed membrane is assumed as

$$\mathbf{p} = \sum_{\alpha=1}^{2} p^\alpha \mathbf{a}_\alpha + p^3 \mathbf{a}_3, \tag{4.1}$$

where p^α, p^3 are regular functions of $\mathbf{x}$, $\alpha = 1, 2$, possibly coincident with the inertial forces per unit area in the dynamic case. By the Cauchy's lemma, for every unit vector $\boldsymbol{\nu}$ belonging to the tangent plane to the surface $\mathcal{B}$ at $\mathbf{x}$, there exists a unique stress tensor field $\mathbf{N} = \mathbf{N}(\mathbf{x})$ such that $\mathbf{n}(\mathbf{x}, \boldsymbol{\nu}) = \mathbf{N}(\mathbf{x})\boldsymbol{\nu}$, where

$$\mathbf{N} = \sum_{\alpha=1}^{2} \mathbf{N}^\alpha \otimes \mathbf{a}_\alpha, \quad \mathbf{N}^\alpha = \mathbf{n}\left(\mathbf{x}, \frac{\mathbf{a}^\alpha}{|\mathbf{a}^\alpha|}\right)|\mathbf{a}^\alpha| \equiv \sum_{\beta=1}^{2} N^{\alpha\beta}\mathbf{a}_\beta, \quad \alpha = 1, 2. \tag{4.2}$$

In particular, on the arc element of the surface with normal $\frac{\mathbf{a}^2}{|\mathbf{a}^2|}$ a force parallel to $\mathbf{a}_2$ is acting and, analogously, on the arc element of surface having normal $\frac{\mathbf{a}^1}{|\mathbf{a}^1|}$ a force parallel to $\mathbf{a}_1$ is acting. Then

$$\mathbf{n}\left(\mathbf{x}, \frac{\mathbf{a}^\alpha}{|\mathbf{a}^\alpha|}\right) = d_\alpha \mathbf{T}_\alpha, \quad \alpha = 1, 2, \tag{4.3}$$

where $\mathbf{T}_\alpha$ is the traction on a *single* thread belonging to the αth coordinate curve (e.g., a force vector parallel to $\mathbf{a}_\alpha$), and d_α is the fiber densities of the αth family threads.

The threads have vanishing shear/bending stiffness and we assume that the magnitude of the force $\mathbf{T}_\alpha$ depends only on the elongation in the direction of the αth coordinate curve, that is

$$\mathbf{T}_\alpha = (\overline{T}_\alpha + A_\alpha \sigma_\alpha)\frac{\mathbf{a}_\alpha}{|\mathbf{a}_\alpha|}, \quad \alpha = 1, 2. \tag{4.4}$$

In the above expression, $\overline{T}_\alpha > 0$ is the tensile pre-stress force acting in the referential configuration $\mathcal{B}_k$; A_α is the area of the cross-section of a single thread belonging to the αth family; and σ_α is the normal stress caused by the deformation of the thread. By (4.2)–(4.4), we have

$$\mathbf{N}^\alpha = d_\alpha(\overline{T}_\alpha + A_\alpha \sigma_\alpha)\frac{|\mathbf{a}^\alpha|}{|\mathbf{a}_\alpha|}\mathbf{a}_\alpha, \quad \alpha = 1, 2, \tag{4.5}$$

or, in controvariant components,

$$N^{11} = d_1(\overline{T}_1 + A_1\sigma_1)\sqrt{\frac{|a^{11}|}{|a_{11}|}}, \tag{4.6}$$

$$N^{22} = d_2(\overline{T}_2 + A_2\sigma_2)\sqrt{\frac{|a^{22}|}{|a_{22}|}}, \tag{4.7}$$

$$N^{12} = N^{21} = 0. \tag{4.8}$$

where $a^{\alpha\alpha} = \mathbf{a}^\alpha \cdot \mathbf{a}^\alpha$, $\alpha = 1, 2$.

Under the assumption of elastic material, we have

$$\sigma_\alpha = E_\alpha \varepsilon_\alpha, \qquad \alpha = 1, 2, \tag{4.9}$$

where $E_\alpha > 0$ is the Young's modulus of the material and ε_α is the elongation measure of the threads belonging to the αth family. By the assumption (2.10) on the displacement field and working within small deformations, the linearized version of ε_α is $\varepsilon_\alpha = \frac{u_\alpha|_\alpha}{A_{\alpha\alpha}}$, $\alpha = 1, 2$, where the covariant derivative $u_\alpha|_\alpha$ of the covariant component u_α with respect to ϑ_α is given by

$$u_\alpha|_\alpha = u_{\alpha,\alpha} - \sum_{\delta=1}^{2} \overline{\Gamma}^\delta_{\alpha\alpha} u_\delta, \qquad \alpha = 1, 2, \tag{4.10}$$

and $\overline{\Gamma}^\delta_{\alpha\beta} = \mathbf{A}_{\alpha,\beta} \cdot \mathbf{A}^\delta$ is the Christoffel symbol defined on the referential configuration $\mathcal{B}_k$. In particular, we have

$$u_1|_1 = u_{1,1}, \quad u_2|_2 = u_{2,2} + \rho u_1 \tag{4.11}$$

and

$$\varepsilon_1 = \frac{u_{1,1}}{(a^2 \cos^2\phi + b^2 \sin^2\phi)}, \qquad \varepsilon_2 = \frac{u_{2,2} + \rho u_1}{\rho^2 (a^2 \sin^2\phi + b^2 \cos^2\phi)}. \tag{4.12}$$

The differential equations of equilibrium can be derived by using the Euler-Cauchy balance force equation on $\mathcal{B}$, using Cauchy's lemma and applying the Divergence Theorem. Under the assumption of smooth tensor and vector fields, we have

$$\begin{cases} \sum_{\alpha=1}^2 N^{\gamma\alpha}|_\alpha + p^\gamma = 0, \quad \text{in } \mathcal{B},\ \gamma = 1, 2, & (4.13) \\ \sum_{\alpha,\beta=1}^2 N^{\beta\alpha} b_{\beta\alpha} + p^3 = 0, \quad \text{in } \mathcal{B}, & (4.14) \end{cases}$$

where

$$N^{\gamma\alpha}|_\alpha = N^{\gamma\alpha}{}_{,\alpha} + \sum_{\delta=1}^{2} N^{\gamma\delta} \Gamma^\alpha_{\delta\alpha} + \sum_{\delta=1}^{2} N^{\delta\alpha} \Gamma^\gamma_{\delta\alpha}, \tag{4.15}$$

$$\Gamma^\gamma_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}^\gamma, \tag{4.16}$$

$$b_{\beta\alpha} = \sum_{\gamma=1}^{2} b^\gamma_\alpha a_{\gamma\beta}, \qquad a_{\gamma\beta} = \mathbf{a}_\gamma \cdot \mathbf{a}_\beta, \qquad b^\gamma_\alpha = -\mathbf{a}_{3,\alpha} \cdot \mathbf{a}^\gamma. \tag{4.17}$$

Finally, by inserting the expressions of the actual thread densities (3.6), (3.7) in (4.6), (4.7), and using the expressions (4.12) of ε_1, ε_2, after neglecting high order terms, we obtain the linearized constitutive equations of the membrane stresses

$$N^{11} = \frac{\bar{d}_1(\overline{T}_1 + A_1\sigma_1)}{ab}\left(1 - u^2_{,2} - \frac{u^1}{\rho} - 2u^1_{,1} - r_2\left(\rho u^2_{,1} + u^2\right)\right)\sqrt{\frac{r_2}{r_1}}, \tag{4.18}$$

$$N^{22} = \frac{\bar{d}_2(\overline{T}_2 + A_2\sigma_2)}{\rho^2 ab}\left(1 - u^1_{,1} - 2\left(u^2_{,2} + \frac{u^1}{\rho}\right) - r_1\left(\frac{u^1_{,2}}{\rho} - u^2\right)\right)\sqrt{\frac{r_1}{r_2}}. \tag{4.19}$$

5. Pre-stress state. The expression of the pre-stress state acting on the referential configuration $\mathcal{B}_k$ of the membrane can be determined by evaluating the expressions (4.18), (4.19) of N^{11}, N^{22} for vanishing displacement field. We have

$$\overline{N}^{11} = \frac{\bar{d}_1\overline{T}_1}{ab}\sqrt{\frac{r_2}{r_1}}, \tag{5.1}$$

$$\overline{N}^{22} = \frac{\bar{d}_2\overline{T}_2}{\rho^2 ab}\sqrt{\frac{r_1}{r_2}}, \quad \left(\overline{N}^{12} = \overline{N}^{21} = 0\right). \tag{5.2}$$

Recalling the expressions (3.1), (3.2) of the fiber densities and the definitions (3.8), (3.9) of the quantities r_1 and r_2, we have

$$\overline{N}^{11} = \frac{C^\rho \overline{T}_1}{\rho ab\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}, \tag{5.3}$$

$$\overline{N}^{22} = \frac{C^\phi \overline{T}_\phi}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}, \tag{5.4}$$

with C^ρ, C^ϕ constants.

The pre-stress field $\overline{N}^{\alpha\beta}$ must satisfy the equilibrium equations under vanishing load, that is

$$\begin{cases} \sum_{\alpha=1}^2 \overline{N}^{\gamma\alpha}|_\alpha = 0, \quad \gamma = 1,2, \qquad \text{in } \mathcal{B}_k, & (5.5)\\ \sum_{\alpha,\beta=1}^2 \overline{N}^{\beta\alpha}\bar{b}_{\beta\alpha} = 0, \qquad \text{in } \mathcal{B}_k, & (5.6)\end{cases}$$

where $\overline{b}_{\beta\alpha}$, $\alpha, \beta = 1, 2$ are the entries of the second fundamental form of the web surface evaluated in the referential (flat) configuration $\mathcal{B}_k$. Since all the $\overline{b}_{\beta\alpha}$'s vanish in $\mathcal{B}_k$, the force equilibrium equation (5.6) in transverse direction is identically satisfied, whereas the in-plane equilibrium equations in (5.5) become

$$\begin{cases} \overline{N}^{\rho\rho}{}_{,\rho} + \dfrac{\overline{N}^{\rho\rho}}{\rho} - \rho\overline{N}^{\phi\phi} = 0, & (5.7) \\ \overline{N}^{\phi\phi}{}_{,\phi} = 0. & (5.8) \end{cases}$$

Note that, hereinafter, we have defined $\overline{N}^{\rho\rho} = \overline{N}^{11}$, $\overline{N}^{\phi\phi} = \overline{N}^{22}$. Moreover, we shall use the notation $\overline{T}_\rho = \overline{T}_1$, $\overline{T}_\phi = \overline{T}_2$, $(\cdot)_{,\rho} = \frac{\partial(\cdot)}{\partial\rho}$, $(\cdot)_{,\phi} = \frac{\partial(\cdot)}{\partial\rho}$.

Equation (5.8) implies

$$\overline{N}^{\phi\phi} = \overline{N}^{\phi\phi}(\rho), \tag{5.9}$$

that is, recalling (5.4),

$$\frac{\overline{T}_\phi(\rho, \phi)}{\sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}} := \overline{\tau}_\phi(\rho), \tag{5.10}$$

where $\overline{\tau}_\rho(\rho)$ is a function to be determined. By (5.10), equation (5.7) can be written as

$$(\rho\overline{N}^{\rho\rho})_{,\rho} = \frac{C^\phi}{ab}\overline{\tau}_\phi(\rho), \tag{5.11}$$

and, therefore,

$$(\rho\overline{N}^{\rho\rho})_{,\rho\phi} = 0, \tag{5.12}$$

which implies

$$\frac{C^\rho}{ab}\frac{\overline{T}_\rho(\rho, \phi)}{\sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi}} = p(\rho) + q(\phi), \tag{5.13}$$

where $p = p(\rho)$, $q = q(\phi)$ are two unknown functions. The above equation shows that the problem of determining the state of pre-stress in the present treatment is underdetermined. In order to investigate on the existence of an equilibrated tensile pre-stress state, we shall assume $q(\phi) \equiv 0$, that is we accept that

$$\frac{\overline{T}_\rho(\rho, \phi)}{\sqrt{a^2 \cos^2\phi + b^2 \sin^2\phi}} := \overline{\tau}_\rho(\rho). \tag{5.14}$$

Inserting (5.10), (5.14) in (5.7), we obtain a single differential equation involving two unknown functions, namely

$$\overline{\tau}'_\rho(\rho) = \xi \overline{\tau}_\phi(\rho), \qquad \text{in } (0, R), \tag{5.15}$$

where $\xi = \frac{C^\phi}{C^\rho} > 0$ and $\overline{\tau}'_\rho(\rho) = \frac{d\overline{\tau}_\rho(\rho)}{d\rho}$.

Following the arguments discussed in Morassi et al. (2017), we shall further introduce an assumption on the circumferential pre-stress $\overline{\tau}_\phi(\rho)$. We can distinguish two main situations, which correspond to the real process followed by the spiders in creating their webs. Referring to Morassi et al. (2017) and to Wirth & Barth (1992) for more details, we recall that in the early stage of the web the spider creates a preliminary family of circumferential threads, called *auxiliary spiral*. The experiments performed by Wirth & Barth (1992) support the hypothesis of proportionality between circumferential and radial pre-stress. According with those observations, we assume

$$\overline{\tau}_\phi(\rho) = k\overline{\tau}_\rho(\rho), \qquad k > 0 \quad \text{constant}. \tag{5.16}$$

Replacing (5.16) in (5.15), and accepting the boundary condition

$$\overline{\tau}_\rho(\rho = R) = \sigma, \qquad \sigma > 0 \quad \text{constant}, \tag{5.17}$$

on the boundary of the elliptic membrane, we obtain

$$\overline{\tau}_\rho(\rho) = \widehat{T} e^{k\xi\rho}, \quad \rho \in [0, R], \tag{5.18}$$

where $\widehat{T} = \sigma e^{-k\xi R} > 0$. Recalling (5.10), (5.14), the tensile pre-stress acting on a single radial ($\overline{T}_\rho$) or elliptical ($\overline{T}_\phi$) thread is

$$\overline{T}_\rho = \widehat{T} e^{k\xi\rho} \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}, \tag{5.19}$$

$$\overline{T}_\phi = k\widehat{T} e^{k\xi\rho} \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}. \tag{5.20}$$

In the second stage of the web construction, the spider removes the auxiliary spiral and adds the threads of the *catching* - or *sticky* - *spiral*. This last configuration is the *finished* web, and arguments discussed in Wirth & Barth (1992) suggest that the tensile pre-stress in the elliptical threads can be assumed approximately constant, namely

$$\overline{\tau}_\phi(\rho) = \mathcal{T} = \text{constant} > 0. \tag{5.21}$$

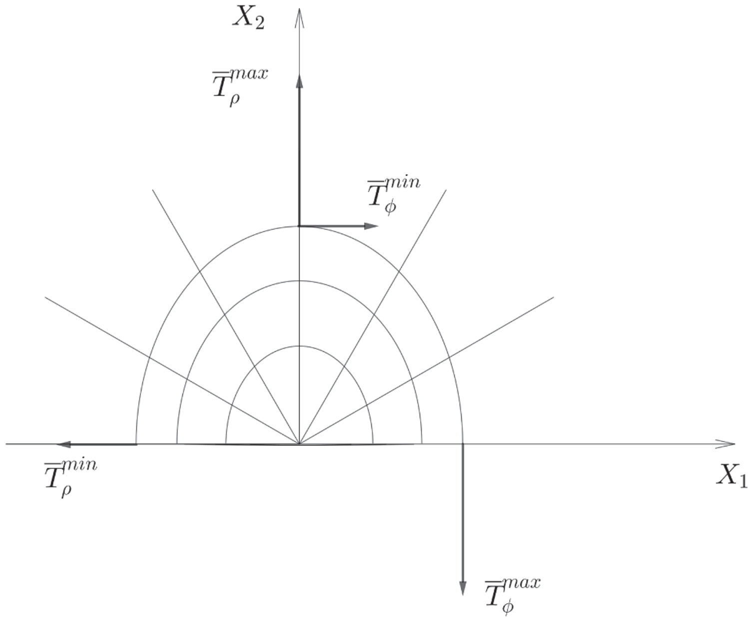


Figure 6: Maximum and minimum pre-stress state values in a spider-web with sticky spiral.

By (5.15) and (5.17), we obtain

$$\overline{\tau}_\rho(\rho) = \widetilde{T} + \xi\mu\rho, \tag{5.22}$$

with $\widetilde{T}$ such that $\widetilde{T} - \xi\mu R > 0$ and, therefore, by (5.10) and (5.14), we have

$$\overline{T}_\rho = (\widetilde{T} + \xi\mathcal{T}\rho)\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}, \tag{5.23}$$

$$\overline{T}_\phi = \mathcal{T}\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}. \tag{5.24}$$

We conclude this section with a couple of remarks on the obtained state of stress in the finished web. Recalling our geometrical assumptions (2.4) and (2.5), from (5.23), (5.24) it turns out that the maximum tensile force in a radial and in an elliptical thread is attained at the points $(X_1 = 0, X_2 = bR)$ $(\overline{T}_\rho^{\max} = \sigma b)$ and $(X_1 = aR, X_2 = 0)$ $(\overline{T}_\phi^{\max} = \mathcal{T}b)$, respectively; see Figure 6. It follows that the maximum tensile force $\overline{T}_\rho$ is attained in the radial thread having maximum length, and this property is in agreement with the need to ensure "uniform" stiffness of the web to transverse loads. Moreover, the maximum value of the traction in the threads belonging to the two families is reached in different points, suggesting an "optimal" distribution of the pre-stress inside the web.

In conclusion, although the a priori assumptions made in deriving the state of pre-stress are rather strong, the final result seems to be reasonable and it will be used in studying the transverse and in-plane dynamic response of the web in the next sections.

6. Transverse motion. By replacing the expressions (4.18) and (4.19) of N^{11} and N^{22} in equation (4.14), after linearization, we obtain the partial differential equation governing the transverse motion of the membrane under the transverse load per unit area p^3 (inertia forces are included):

$$\frac{C^\rho \overline{T}_\rho}{\rho ab\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} w_{,\rho\rho} + \frac{C^\phi \overline{T}_\phi}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} w_{,\phi\phi} + \\ + \frac{C^\phi \overline{T}_\phi}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} \rho w_{,\rho} + p^3 = 0, \tag{6.1}$$

or, equivalently, recalling (5.10) and (5.14),

$$\frac{C^\rho}{\rho ab}\overline{\tau}_\rho w_{,\rho\rho} + \frac{C^\phi}{\rho^2 ab}\overline{\tau}_\phi (w_{,\phi\phi} + \rho w_{,\rho}) + p^3 = 0, \tag{6.2}$$

where $\overline{\tau}_\phi$, $\overline{\tau}_\rho$, are given by (5.16)-(5.18), (5.21)-(5.22) for the web with auxiliary or catching spiral, respectively.

In the sequel we shall investigate on a special case of (6.2), namely the undamped transverse free vibrations of the elliptical membrane supported at the boundary, i.e.,

$$u^3(R,\phi,t) = 0, \qquad (\phi,t) \in [0,2\pi] \times [0,\infty). \tag{6.3}$$

In this case, the function p^3 in (6.2) coincides with the surface density of the out-of-plane inertia forces. Denoting by m_ρ and m_ϕ the uniform linear mass density of the radial and elliptical threads, respectively, the surface mass density γ of the continuum model can be determined as

$\gamma = \frac{dm}{dA}$, where $dA = |\mathbf{A}_1 \times \mathbf{A}_2|\, d\rho\, d\phi = \rho ab\, d\rho\, d\phi$ is the elementary area in $\mathcal{B}_k$ and dm is the elementary mass of the threads lying in dA. It turns out that

$$\gamma(\rho,\phi) = \frac{C^\rho}{\rho ab} m_\rho \sqrt{a^2\cos^2\phi + b^2\sin^2\phi} + \frac{C^\phi}{ab} m_\phi \sqrt{a^2\sin^2\phi + b^2\cos^2\phi} \tag{6.4}$$

and, therefore, the transverse motion equation becomes

$$\frac{C^\rho}{\rho ab}\overline{\tau}_\rho u^3_{,\rho\rho} + \frac{C^\phi}{\rho^2 ab}\overline{\tau}_\phi (u^3_{,\phi\phi} + \rho u^3_{,\rho}) - \gamma(\rho,\phi) u^3_{,tt} = 0, \tag{6.5}$$

for $(\phi,\rho,t) \in (0,2\pi)\times(0,R)\times(0,\infty)$. Setting

$$u^3 = w(\rho,\phi) y(t), \tag{6.6}$$

we can separate the variables (ρ,ϕ) from the time variable t, obtaining

$$y'' + \lambda y = 0, \qquad t > 0, \tag{6.7}$$

and, using (5.15),

$$(\overline{\tau}_\rho w_{,\rho})_{,\rho} + \lambda ab \widetilde{\gamma} w = -\frac{g}{\rho} w_{,\phi\phi}, \tag{6.8}$$

where $\widetilde{\gamma} = \frac{\rho}{C^\rho}\gamma$, $\lambda \in \mathbb{R}^+$ is the eigenvalue to be determined and

$$\begin{cases} k\xi\overline{\tau}_\rho & \text{(unfinished web)}, \quad (6.9) \\ \xi\mu & \text{(finished web)}. \quad (6.10) \end{cases}$$

The expression (6.4) of the mass density γ prevents the separation between the radial variable ρ and the angular variable ϕ. Therefore, in the sequel we only provide estimates, from below and from above, of the eigenvalues of (6.8) under the boundary condition (6.3). It is easy to show that

$$\widetilde{\gamma}^-(\rho) \equiv a(m_\rho + \xi m_\phi \rho) \le ab\widetilde{\gamma} \le b(m_\rho + \xi m_\phi \rho) \equiv \widetilde{\gamma}^+(\rho), \tag{6.11}$$

and it should be noticed that the variables ρ and ϕ can be separated in (6.8) when $ab\widetilde{\gamma}$ is replaced either by $\widetilde{\gamma}^-$ or by $\widetilde{\gamma}^+$. Let us consider, for example, the coefficient $\widetilde{\gamma}^+$. We can look for a solution to (6.8) (with $ab\widetilde{\gamma}$ replaced by $\widetilde{\gamma}^+$) of the form

$$w(\rho,\phi) = u(\rho)\Phi(\phi), \tag{6.12}$$

where, by regularity of w, $\Phi(\phi)$ is a non trivial solution of the eigenvalue problem

$$
\begin{cases}
\Phi'' + \nu^2 \Phi = 0, & (6.13)\\
\Phi(0) = \Phi(2\pi), & (6.14)\\
\Phi'(0) = \Phi'(2\pi), & (6.15)
\end{cases}
$$

and $u(\rho)$ solves

$$
(\overline{\tau}_\rho u')' + \lambda^+ \widetilde{\gamma}^+ u = \frac{\nu^2}{\rho} g u, \qquad \rho \in (0, R), \tag{6.16}
$$

where the function $g = g(\rho)$ is defined by (6.9)-(6.10). It is easy to show that the eigenpairs to (6.13)-(6.15) are

$$
\nu_n^2 = n^2, \quad \Phi_n(\phi) = A\cos(n\phi) + B\sin(n\phi), \quad n = 0, 1, 2, ... \tag{6.17}
$$

If $n = 0$, then $\nu_0 = 0$ and $\Phi_0(\phi)$ is a non vanishing constant. The corresponding eigenfunctions w are functions of the variable ρ only, and can be determined by solving the problem

$$
\begin{cases}
(\overline{\tau}_\rho u_0')' + \lambda_0^+ \widetilde{\gamma}^+ u_0 = 0, \quad \rho \in (0, R), & (6.18)\\
u_0(R) = 0, & (6.19)\\
u_0'(0) = 0, & (6.20)
\end{cases}
$$

where in deriving the boundary condition (6.20) the absence of a concentrated transverse force acting in O has been taken into account. Problem (6.18)–(6.20) admits simple real eigenvalues $\{\lambda_{0,j}^+\}_{j=1}^\infty$ such that

$$
0 < \lambda_{0,1}^+ < \lambda_{0,2}^+ < ..., \quad \lim_{j\to\infty} \lambda_{0,j}^+ = +\infty. \tag{6.21}
$$

When $n \geq 1$, the eigenfunctions $u(\rho)$ in (6.12) can be determined by solving

$$
\begin{cases}
(\overline{\tau}_\rho u')' + \lambda_n^+ \widetilde{\gamma}^+ u = \frac{n^2}{\rho} g u, \quad \rho \in (0, R), & (6.22)\\
u(R) = 0, & (6.23)\\
u(0) = 0. & (6.24)
\end{cases}
$$

For every n, $n \geq 1$, the eigenvalues of the above problem will be indicated as $\{\lambda_{n,m}^+\}_{m=1}^\infty$, with

$$
0 < \lambda_{n,1}^+ < \lambda_{n,2}^+ < ..., \quad \lim_{m\to\infty} \lambda_{n,m}^+ = +\infty. \tag{6.25}
$$

Note that the end condition $u(0) = 0$ guarantees for finite values of the strain energy associated with the transversal deformation $u = u(\rho)$.

In conclusion, the elliptic membrane with mass density $\widetilde{\gamma}^+$ as in (6.11) has the sequence of eigenvalues $\{\lambda^+_{n,m}\}_{m=1}^{\infty}$, $n = 0, 1, 2, ...$ By repeating the above analysis with $\widetilde{\gamma}^-$, we obtain the eigenvalues $\{\lambda^-_{n,m}\}_{m=1}^{\infty}$, $n = 0, 1, 2, ...$ Finally, by a monotonicity theorem, see Courant & Hilbert (1965), we can estimate from above and from below the eigenvalues of (6.8). More precisely, after a suitable reordering, every eigenvalue $\{\lambda_{n,m}\}$ of (6.8) is such that

$$\lambda_{n,m}(\widetilde{\gamma}^+) \leq \lambda_{n,m}(\widetilde{\gamma}) \leq \lambda_{n,m}(\widetilde{\gamma}^-). \tag{6.26}$$

The accuracy of the above bounds clearly depends on the ratio $\frac{b}{a}$, see (6.11). However, for $\frac{b}{a} \in [1.1, 1.3]$ it is expected that $\lambda_{n,m}(\widetilde{\gamma}^+)$, $\lambda_{n,m}(\widetilde{\gamma}^-)$ offer a good approximation of the actual eigenvalue $\lambda_{n,m}(\widetilde{\gamma})$.

7. In-plane deformation. In this section we write the equations governing the in-plane mechanical behavior of the elliptic membrane. A complete study of the in-plane problem is outside the goals of the present note, and it will be the object of future investigation. Using the constitutive equations (4.18), (4.19) for $N^{\rho\rho}$, $N^{\phi\phi}$, respectively, within the equilibrium equations (4.13), recalling (5.10), (5.14) and passing to contravariant components, after linearization we obtain

$$\begin{aligned}
&\frac{C^{\phi}\overline{\tau}_{\phi}}{\rho^2 ab}[u^{\rho}_{,\phi\phi} - \rho u^{\phi}_{,\phi} - r_2(\rho^2 u^{\phi}_{,\rho} + \rho u^{\phi}) + r_1(u^{\rho}_{,\phi} - \rho u^{\phi})] \\
&- \frac{C^{\rho}\overline{\tau}_{\rho}}{\rho ab} r_2(\rho u^{\phi}_{,\rho\rho} + 2u^{\phi}_{,\rho}) + \frac{C^{\rho}A_{\rho}E_{\rho}[u^{\rho}_{,\rho\rho} + r_2(\rho u^{\phi}_{,\rho\rho} + 2u^{\phi}_{,\rho})]}{\rho ab\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} + \\
&- \frac{C^{\phi}A_{\phi}E_{\phi}[u^{\rho} + \rho u^{\phi}_{,\phi} + r_1(u^{\rho}_{,\phi} - \rho u^{\phi})]}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} + \gamma(\rho,\phi)u^{\rho}_{,tt} = 0,
\end{aligned} \tag{7.1}$$

$$
\begin{aligned}
&\frac{C^\phi \overline{\tau}_\phi}{\rho^2 ab}\left[\rho u^\phi_{,\rho} + u^\phi + \frac{a^2b^2}{\rho}\frac{(u^\rho_{,\phi} - \rho u^\phi)}{(a^2\sin^2\phi + b^2\cos^2\phi)^2}\right] + \\
&- \frac{C^\phi \overline{\tau}_\phi}{\rho^2 ab}\left[r_1^2\left(\frac{u^\rho_{,\phi}}{\rho} - u^\phi\right) - \frac{r_1}{\rho}(u^\rho_{,\phi\phi} - \rho u^\phi_{,\phi})\right] + \\
&+ \frac{C^\phi A_\phi E_\phi}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}\left[\frac{2}{\rho}u^\rho_{,\phi} - u^\phi + u^\phi_{,\phi\phi} + \frac{r_1}{r_2}\left(u^\phi - \frac{u^\rho_{,\phi}}{\rho}\right)\right] + \\
&+ \frac{C^\phi A_\phi E_\phi}{\rho^2 ab\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}\left[\frac{r_1}{\rho}(u^\rho + u^\rho_{,\phi\phi} + 3r_1(u^\rho_{,\phi} - \rho u^\phi))\right] + \\
&+ \frac{C^\rho \overline{\tau}_\rho}{\rho ab}\left(u^\phi_{,\rho\rho} + \frac{2}{\rho}u^\phi_{,\rho}\right) + \gamma(\rho,\phi)u^\phi_{,tt} = 0,
\end{aligned}
\tag{7.2}
$$

where A_ρ, A_ϕ is the area of the cross-section of a single radial and elliptical thread, respectively, and E_ρ, E_ϕ are the Young's modulus of the material forming the two families of fibers. Equations (7.1) and (7.2) express the dynamic equilibrium for undamped free infinitesimal vibrations in radial and angular direction, respectively.

8. Conclusions. The formulation of mechanical models of the dynamic response of spider webs has raised an increasing interest in recent years, due to its implications in the study of the spider behavior. Morassi, Soler and Zaera (2017) proposed a continuous model of pre-tensed structured membrane of orb-webs. The model was developed under the assumptions of axial symmetry and small deformations. The main purpose of this note was to generalize the above approach to spider webs of elliptical shape. The extension required the introduction of appropriate a priori hypotheses for determining an admissible pre-tension state in the referential configuration. The present work should be considered as a preliminary step towards the study of more realistic spider web geometries having a single symmetry axis.

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