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Exact determination of beams with given buckling loads

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Abstract. We present an analytical procedure for the determination of Euler-Bernoulli beams which have given values of the first N buckling loads. The result is valid for pinned-pinned end conditions and for beams with regular bending stiffness. The analysis is based on a reduction of the buckling problem to an eigenvalue problem for a vibrating string, and uses recent results on the exact construction of Sturm-Liouville operators with prescribed natural frequencies.

Key-words. Buckling loads, beams, Darboux Lemma, quasi-isospectral operators, inverse problems.

1. Introduction. In the recent paper (Caliò et al., 2011) the authors have shown how to construct families of Euler-Bernoulli beams which have exactly the same infinite sequence of buckling loads of a given beam under a specified set of end conditions. These beams are called *isobuckling beams*.

The research developed in (Caliò et al., 2011) left an important question unsolved, namely: can we construct an Euler-Bernoulli beam which has exactly given values of the first N buckling loads?

In this note we give a positive answer to the above question and, under suitable assumptions, we present a constructive explicit procedure for solving the inverse problem.

Our result holds for beams simply supported at the ends subject to constant compressive axial load and with regular bending stiffness coefficient. The analysis is based on a reduction of the buckling problem to

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an equivalent eigenvalue problem for a class of fixed-fixed strings, and adapts recent results on the exact construction of second-order Sturm-Liouville operators in canonical form with prescribed natural frequencies, see (Morassi, 2015). In particular, the key mathematical tool relies on a classical lemma by Darboux (Darboux, 1882), which allows to explicitly construct families of Sturm-Liouville operators that share all the eigenvalues of a given Sturm-Liouville operator, with the exception of a single eigenvalue which is free to move in a prescribed interval. These operators are called *quasi-isospectral operators*. Finally, the analysis shown in (Morassi, 2015) is used to determine strings corresponding to the quasi-isospectral Sturm-Liouville operators and, ultimately, to find beams quasi-isobuckling to a given beam.

2. Elastic buckling of a beam and an equivalent string problem. Consider a thin straight elastic beam under constant compressive axial load P, $P > 0$. The buckling problem is governed by the Euler-Bernoulli-Kirchhoff equation (see (Love, 1944))

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2v(x)}{dx^2}\right) + P\frac{d^2v(x)}{dx^2} = 0, \quad x \in (0, L), \tag{2.1}$$

where $v = v(x)$ is the transverse displacement of the beam axis at the cross-section of abscissa x evaluated with the principal plane of bending. In equation (2.1), E is the Young's modulus of the material, $E = const. > 0$, and $I = I(x)$ is the second moment of the cross-sectional area about a principal axis through the centroid of the cross-section. We shall be concerned with beams for which $I(x)$ is a strictly positive, twice continuously differentiable function of x in $[0, L]$, e.g.

$$I(x) \geq I_0 > 0, \quad x \in [0, L], \quad I \in C^2([0, L]). \tag{2.2}$$

Let us assume that the beam has Pinned-Pinned (P-P) ends. The buckling problem consists in solving the eigenvalue problem

$$\begin{cases} \frac{d^2}{dx^2}\left(I(x)\frac{d^2v(x)}{dx^2}\right) + \lambda^2\frac{d^2v(x)}{dx^2} = 0, \quad x \in (0, L), & (2.3)\\ v(0) = \frac{d^2v(0)}{dx^2} = 0, & (2.4)\\ v(L) = \frac{d^2v(L)}{dx^2} = 0, & (2.5) \end{cases}$$

where

$$\lambda^2 = \frac{P}{E}. \tag{2.6}$$

Under the above assumptions, there exists an infinite sequence of buckling loads $\{P_m = \lambda_m^2 E\}_{m=1}^{\infty}$, with

$$0 < P_1 < P_2 < ..., \quad \lim_{m\to\infty} P_m = \infty, \tag{2.7}$$

such that (2.3)–(2.5) have a non-trivial solution $v_m = v_m(x)$, $m \geq 1$. This sequence is the *buckling spectrum* of the Pinned-Pinned beam and we write

$$\{\lambda_m^2\}_{m=1}^{\infty} = \mathtt{BSp}(I(x); P-P). \tag{2.8}$$

The following proposition states the equivalence between the eigenvalue problem (2.3)–(2.5) and the free vibration problem for a family of taut strings.

Proposition 2.1:
If $\{\lambda^2, v(x)\}$ *is an eigenpair of* (2.3)–(2.5) *with* $I = I(x)$ *satisfying* (2.2), *then* $\{\lambda^2, v(x)\}$ *is an eigenpair of*

$$\begin{cases} \frac{d^2v(x)}{dx^2} + \lambda^2\rho(x)v(x) = 0, \quad x \in (0, L), & (2.9)\\ v(0) = 0 = v(L), & (2.10) \end{cases}$$

with

$$\rho(x) = \frac{1}{I(x)}, \quad x \in [0, L]. \tag{2.11}$$

Viceversa, if $\{\lambda^2, v(x)\}$ *is an eigenpair of* (2.9)–(2.10), *then* $\{\lambda^2, v(x)\}$ *is an eigenpair of* (2.3)–(2.5).

The eigenvalue problem (2.9)–(2.10) describes the free, infinitesimal, transverse vibration of amplitude $v = v(x)$ of a string with frequency λ and mass density $\rho = \rho(x)$, $\rho \in C^2([0, L])$ and $\rho(x) \geq \rho_0 > 0$ in $[0, L]$. The string is pulled with unit tension, has length L and is fixed at both ends. A proof of Proposition (2.1) is presented in (Caliò et al., 2011) (Proposition 1).

3. Construction of beams with given buckling loads. Let n, $n \geq 1$, be given. The key step of our method is based on the explicit construction of a new P-P beam quasi-isobuckling to the given beam, that is a beam $I = I(x)$ having the same buckling loads as the given beam $\widehat{I} = \widehat{I}(x)$, with the exception of the nth buckling load. In fact, by keeping fixed all the eigenvalues λ_m^2 with $m \neq n$ and moving the nth eigenvalue λ_n^2 to the desired value, say $\widetilde{\lambda}_n^2$, and using repeatedly the procedure, after N steps we will construct a beam with the first N given eigenvalues $\{\widetilde{\lambda}_m^2\}_{m=1}^N$, and the construction is completed.

The main steps of the construction of P-P beams $I = I(x)$ quasi-isobuckling to a given P-P beam $\widehat{I} = \widehat{I}(x)$ are the following.

Step 1. The string eigenvalue problem (2.9)–(2.10) is reduced to Sturm-Liouville canonical form with Schrödinger potential $\widehat{q}$ (see Section 3.1).

Step 2. The Darboux Lemma (see Appendix) is used to construct explicit families of Schrödinger potentials q quasi-isospectral to the initial potential $\widehat{q}$ (see Section 3.2).

Step 3. The Darboux Lemma is applied once more in iterate form to determine string mass densities corresponding to the quasi-isospectral potentials q (see Section 3.3).

Step 4. Finally, the equivalence stated in Proposition 2.1 is used to find P-P beams $I = I(x)$ quasi-isobuckling to the initial P-P beam $\widehat{I} = \widehat{I}(x)$ (see Section 3.4).

We shall analyze Steps 1-4 in the following subsections.

3.1 Reduction to canonical form. Suppose that a P-P beam $\widehat{I} = \widehat{I}(x)$, satisfying conditions (2.2), is given. The buckling spectrum of this beam is $\{\widehat{\lambda}_m^2\}_{m=1}^\infty = \mathtt{BSp}(\widehat{I}(x); P-P)$. Denote by $\{\widehat{\rho}(x)\}$ the corresponding Fixed-Fixed (F-F) string as defined in Proposition 2.1, with spectrum $\{\widehat{\lambda}_m^2\}_{m=1}^\infty = \mathtt{Sp}(\widehat{\rho}(x); F-F)$. The Liouville transformation

$$\xi(x) = \frac{1}{\widehat{p}} \int_0^x (\widehat{\rho}(s))^{1/2} ds, \quad \widehat{p} = \int_0^L (\widehat{\rho}(s))^{1/2} ds, \tag{3.1}$$

$$y(\xi) = \widehat{a}(\xi) v(x), \quad \widehat{a}^4(\xi) = \frac{L^2}{\widehat{p}^2} \widehat{\rho}(x), \tag{3.2}$$

reduces the eigenvalue problem (2.9)–(2.10) (with ρ replaced by $\widehat{\rho}$) for $\{\widehat{\lambda}^2, v(x)\}$ to the Sturm-Liouville canonical form

$$\begin{cases} \frac{d^2y(\xi)}{d\xi^2} + \widehat{\mu}y(\xi) = \widehat{q}(\xi)y(\xi), \quad \xi \in (0,1), & (3.3) \\ y(0) = 0 = y(1), & (3.4) \end{cases}$$

where the eigenvalue $\widehat{\mu}$ and the potential $\widehat{q}(\xi)$, $\widehat{q} \in C^0([0,1])$, are defined as

$$\widehat{\mu} = \widehat{p}^2\widehat{\lambda}^2, \quad \widehat{q}(\xi) = \frac{1}{\widehat{a}(\xi)}\frac{d^2\widehat{a}(\xi)}{d\xi^2}\ , \quad \xi \in (0,1). \tag{3.5}$$

3.2 Quasi-isospectral potentials. Following the analysis developed in (Pöschel & Trubowitz, 1987), it is possible to explicitly construct families of Sturm-Liouville operators $L = -\frac{d^2}{d\xi^2} + q(\xi)$, with potential $q(\xi)$ quasi-isospectral to the potential $\widehat{q}(\xi)$ under Dirichlet end conditions. The analysis is based on the Darboux Lemma described in the Appendix. Here, we simply recall the main result. Let us introduce some notation. Let n, $n \geq 1$, be a given number and let $t \in \mathbb{R}$ be such that

$$\mu_{n-1}(\widehat{q}) < \mu_n(\widehat{q}) + t < \mu_{n+1}(\widehat{q}), \tag{3.6}$$

with $\mu_0(\widehat{q}) = 0$. Denote by δ_{ij} the Kronecker symbol. For $\mu \in \mathbb{C}$, let $y_i = y_i(\xi, \widehat{q}, \mu)$, $i = 1, 2$, be the solution to the initial value problem

$$\begin{cases} y_i'' + \mu y_i = \widehat{q}y_i, \qquad x \in (0,1), & (3.7) \\ y_i(0) = \delta_{i1}, & (3.8) \\ y_i'(0) = \delta_{i2}, & (3.9) \end{cases}$$

and denote by $w_n = w_n(\xi, \widehat{q}, \mu)$ the solution to

$$\begin{cases} w_n'' + \mu w_n = \widehat{q}w_n, \qquad \xi \in (0,1), & (3.10) \\ w_n(0) = 1, & (3.11) \\ w_n(1) = y_1(1, \mu_n, \widehat{q}), & (3.12) \end{cases}$$

for $\mu \neq \mu_n$ (note that the function w_n has a removable singularity at $\mu = \mu_n$). Let

$$\omega_n(\xi, \widehat{q}, \mu) = w_n(\xi, \widehat{q}, \mu)\frac{dz_n(\xi, \widehat{q})}{d\xi} - \frac{dw_n(\xi, \widehat{q}, \mu)}{d\xi}z_n(\xi, \widehat{q}) \tag{3.13}$$

where z_n is the nth eigenfunction of (3.3)–(3.4) and $(\cdot)' = \frac{d(\cdot)}{d\xi}$. For every $\widehat{q} \in C^0([0,1])$, the function $\omega_n = \omega_n(\xi, \widehat{q}, \mu)$, $n \geq 1$, is a continuous and strictly positive function on $[0,1] \times (\mu_{n-1}(\widehat{q}), \mu_{n+1}(\widehat{q}))$. Moreover, ω_n is a C^2-function of the variable ξ in $[0,1]$; see (Pöschel & Trubowitz, 1987). We define $w_{n,t} = w(\xi, \widehat{q}, \mu_n + t)$ and $\omega_{n,t} = \omega(\xi, \widehat{q}, \mu_n + t)$.

Under the above notation, for every given n, $n \geq 1$, and t satisfying (3.6), it is possible to prove that the potential

$$q(\xi) = \widehat{q}(\xi) - 2\frac{d^2}{d\xi^2}(\ln \omega_{n,t}(\xi)) \tag{3.14}$$

has all the same eigenvalues of the potential $\widehat{q}(\xi)$, with the exception of the nth eigenvalue, which takes the value $\mu_n(q) = \mu_n(\widehat{q}) + t$. Moreover, the eigenfunctions $\{k_{m,t}\}_{m=1}^{\infty}$ associated to $q(\xi)$ have the following explicit expressions

$$k_{m,t} = z_m - t\frac{w_{n,t}}{\omega_{n,t}} \int_0^{\xi} z_m(s) z_n(s) ds, \quad \text{for } m \geq 1,\ m \neq n, \tag{3.15}$$

$$k_{n,t} = \frac{z_n}{\omega_{n,t}}. \tag{3.16}$$

3.3 Quasi-isospectral strings. The eigenvalues $\{\widehat{\mu}_m\}$ of (3.3)–(3.4) have the asymptotic form

$$\widehat{\mu}_m = (m\pi)^2 + \widehat{O}(1), \quad \text{as } m \to \infty, \tag{3.17}$$

with $\widehat{O}(1)$ bounded quantity as $m \to \infty$. Therefore, if the two strings $\{\widehat{\rho}(x)\}$ and $\{\rho(x)\}$ are quasi-isospectral, i.e. $\widehat{\lambda}_m^2 = \lambda_m^2$ for every $m \neq n$, where $n \geq 1$ is a given number, then, for m large,

$$\widehat{p}^2 \widehat{\lambda}_m^2 = (m\pi)^2 + \widehat{O}(1), \quad p^2 \lambda_m^2 = (m\pi)^2 + O(1), \tag{3.18}$$

so that

$$\widehat{p}^2 = p^2. \tag{3.19}$$

Now, to find a supported string $\{\rho(x)\}$ quasi-isospectral to a given supported string $\{\widehat{\rho}(x)\}$, we must preliminarily find a function $a = a(\xi)$ corresponding to the new quasi-isospectral potential $q = q(\xi)$ given by (3.14), that is

$$\frac{d^2 a(\xi)}{d\xi^2} = q(\xi) a(\xi), \tag{3.20}$$

with $a = a(\xi)$ of one-sign in $[0,1]$. A double application of the Darboux Lemma yields the following explicit expression for a:

$$a(\xi) = \widehat{a}(\xi) - t\frac{w_{n,t}(\xi)}{\mu_n \omega_{n,t}(\xi)}[z_n, \widehat{a}](\xi), \qquad n \geq 1, \tag{3.21}$$

see (Morassi, 2015) for details. In particular, it is possible to prove that $a = a(\xi)$ given by (3.21) is a C^2-function of one sign in $[0,1]$ for every t satisfying (3.6).

To complete the construction of quasi-isospectral strings, we reverse the Liouville transformation (3.1)–(3.2), namely

$$x(\xi) = \frac{L}{K}\int_0^{\xi} \frac{ds}{a^2(s)}, \quad K = \int_0^1 \frac{ds}{a^2(s)}, \tag{3.22}$$

$$v(x) = \frac{y(\xi)}{a(\xi)}, \quad \rho(x) = \frac{\widehat{p}^2 K^2}{L^2} a^4(\xi), \tag{3.23}$$

and the Sturm-Liouville eigenvalue problem (3.3)–(3.4) (with $\widehat{q}(\xi)$ replaced by $q(\xi)$) is transformed back into the string eigenvalue problem

$$\begin{cases} \frac{d^2 v(x)}{dx^2} + \lambda^2 \rho(x) v(x) = 0, \quad x \in (0, L), & (3.24)\\ v(0) = 0 = v(L). & (3.25)\end{cases}$$

Therefore, the two strings $\{\widehat{\rho}(x)\}$, $\{\rho(x)\}$ of equal length L, having fixed-fixed end conditions and pulled by unit tension, are quasi-isospectral. More precisely, given a number n, $n \geq 1$, we have $\lambda_m^2(\widehat{\rho}(x)) = \lambda_m^2(\rho(x))$ for every $m \geq 1$, $m \neq n$, and the nth eigenvalue $\lambda_n^2(\rho(x))$ is connected with $\lambda_n^2(\widehat{\rho}(x))$ via (3.6).

3.4 Constructing beams with a given finite set of buckling loads. In this section we shall complete the proof of the main result of the paper. The analysis follows the lines of the corresponding analysis developed in (Morassi, 2015) (Section 5) for the determination of families of beams with a given set of natural frequencies.

Let us consider a P-P beam with $I_0 = I_0(x)$ and buckling loads $\{\lambda_m^2(I_0)\}_{m=1}^{\infty}$ (e.g., eigenvalues of (2.3)–(2.5) with $I(x)$ replaced by $I_0(x)$). Starting from this P-P beam, we wish to construct a new beam P-P having prescribed values of the first N, $N \geq 1$, buckling loads $\{\widetilde{\lambda}_m^2\}_{m=1}^{N}$, with

$$0 < \widetilde{\lambda}_1^2 < \widetilde{\lambda}_2^2 < ... < \widetilde{\lambda}_N^2 \ . \tag{3.26}$$

Following the analysis of the previous sections, starting from the beam $I_0(x)$ we can construct a new beam $I_1(x)$ so that $\lambda_m^2(I_1) = \lambda_m^2(I_0)$ for $m \geq 2$, and $\lambda_1^2(I_1)$ coincides with the desired value $\widetilde{\lambda}_1^2$. More precisely, denoting by $a_0(\xi)$ the function $\widehat{a}(\xi)$ appearing in (3.2) (and corresponding to the initial beam $I_0(x)$), the function $a_1 = a_1(\xi)$ associated to the new beam $I_1(x)$ is given by (3.21):

$$a_1(\xi) = a_0(\xi) - t\frac{w_{1,t}(\xi)}{\mu_1(I_0)\omega_{1,t}(\xi)}[z_1(I_0), a_0](\xi) \tag{3.27}$$

where the functions $w_{1,t}(\xi)$, $\omega_{1,t}(\xi)$ are defined in (3.10)–(3.12), (3.13), respectively, with $\widehat{q}(\xi)$ replaced by $\widehat{q}_0(\xi) = \frac{1}{a_0(\xi)}\frac{d^2 a_0(\xi)}{d\xi^2}$. Moreover, μ_m and λ_m are linked as in (3.5), and t satisfies (3.6). If $\widetilde{\mu}_1 < \mu_2(I_0)$, then we can determine t, say $t = t_1$, such that $\mu_1(I_1) = \widetilde{\mu}_1$. The new beam $I_1(x)$ has buckling loads (or eigenvalues) $\{\widetilde{\lambda}_1^2, \lambda_2^2(I_0), \lambda_3^2(I_0), ...\}$, with $0 < \widetilde{\lambda}_1^2 < \lambda_2^2(I_0) < \lambda_3^2(I_0) < ...$, and can be used as starting point for the next step of the construction.

By repeating the above arguments, and provided that $\widetilde{\mu}_2 < \mu_3(I_0)$, we can modify I_1 so as to keep $\lambda_m^2(I_1)$ fixed for $m \neq 2$ and move $\lambda_2^2(I_1)$ to the desired value $\widetilde{\lambda}_2^2$, by taking

$$a_2(\xi) = a_1(\xi) - t_2\frac{w_{2,t_2}(\xi)}{\mu_2(I_1)\omega_{2,t_2}(\xi)}[z_2(I_1), a_1](\xi), \tag{3.28}$$

where

$$t_2 = \widetilde{\mu}_2 - \mu_2(I_0). \tag{3.29}$$

The buckling loads of the P-P beam $I_2(x)$ (associated to $a_2(\xi)$) are $\{\widetilde{\lambda}_1^2, \widetilde{\lambda}_2^2, \lambda_3^2(I_0), \lambda_4^2(I_0), ...\}$. By using repeatedly this procedure, after N steps we construct a beam with coefficient $I_N(x)$ such that

$$\lambda_m^2(I_N) = \widetilde{\lambda}_m^2, \qquad \text{for } 1 \leq m \leq N, \tag{3.30}$$

and the construction is completed. Clearly, the choice of the initial beam $I_0(x)$ is restricted by the conditions

$$\widetilde{\lambda}_1^2 < \lambda_2^2(I_0), \quad \widetilde{\lambda}_2^2 < \lambda_3^2(I_0), \quad ..., \quad \widetilde{\lambda}_{N-1}^2 < \lambda_N^2(I_0), \quad \widetilde{\lambda}_N^2 < \lambda_{N+1}^2(I_0), \tag{3.31}$$

which allow to determine uniquely the numbers t_1, t_2, ..., t_N by expressions analogous to equation (3.29).

We notice that the above construction is not unique, since the flow from the initial beam I_0 to a beam with prescribed values of the first N buckling loads depends on the particular order chosen to move every individual eigenvalue to the target value. As a consequence, the conditions (3.31) on the initial beam I_0 may change depending on the sequence of eigenvalue shifts.

Finally, we remark that previous arguments can be adapted to cover other sets of end conditions. In fact, by Proposition 2 of (Caliò et al., 2011), the equivalence between the buckling problem for beams and the eigenvalue problem for strings stated in Proposition 2.1 can be extended to situations in which the beam, for example, has left end pinned and right end with a sliding constraint, e.g., $\frac{dv}{dx}(L) = 0$ and $\frac{d}{dx}\left(I\frac{d^2v}{dx^2}\right)(L) = 0$. The correspondence will link pinned and sliding end of the beam to fixed and free end of the string, respectively.

4. Conclusions. In this paper we have considered the problem of constructing Euler-Bernoulli beams with prescribed values of the first N buckling loads, under a specified set of boundary conditions. The analysis is based on the fact that the buckling problem for a pinned-pinned beam is equivalent to the eigenvalue problem for a fixed-fixed vibrating string. The key point of the procedure is the determination of quasi-isospectral strings, that is strings with different mass density which have the same spectrum as the original string, with the exception of a given eigenvalue which is free to move in a prescribed interval. Quasi-isospectral systems follow from suitable application of a Darboux Lemma, after reduction of the string equation to canonical Sturm-Liouville form. The reconstruction procedure needs the specification of an initial beam whose the buckling loads must satisfy certain interlacing conditions with the assigned buckling loads. A theoretical aspect worth of investigation, and still open, is the characterization of the set of beams that could be chosen as starting point of the procedure.

5. Appendix. In this appendix we recall the Darboux Lemma.

Lemma 5.1 ((Darboux, 1882)):
Let μ be a real number, and suppose $g \equiv g(\xi)$ is a non-trivial solution

of the Sturm-Liouville equation

$$-g'' + \widehat{q}g = \mu g, \tag{5.1}$$

with continuous potential $\widehat{q} \equiv \widehat{q}(\xi)$*. If* f *is a non-trivial solution of*

$$-f'' + \widehat{q}f = \lambda f \tag{5.2}$$

and $\lambda \neq \mu$*, then*

$$y = \frac{1}{g}[g, f] \equiv \frac{1}{g}(gf' - g'f) \tag{5.3}$$

is a non-trivial solution of the Sturm-Liouville equation

$$-y'' + \check{q}y = \lambda y, \tag{5.4}$$

where

$$\check{q} = \widehat{q} - 2(\ln(g(\xi))''. \tag{5.5}$$

Moreover, the general solution of the equation

$$-y'' + \check{q}y = \mu y \tag{5.6}$$

is

$$y = \frac{1}{g}\left(b_1 + b_2 \int_0^\xi g^2(s)ds\right), \tag{5.7}$$

where b_1 *and* b_2 *are arbitrary constants. In particular,* $y = \frac{1}{g}$ *is a solution of* (5.6)*.*

It should be noted that if g vanishes in $[0, 1]$, then equation (5.4) is understood to hold between the roots of g. These singular situations disappear by applying the Darboux Lemma twice.

Bibliografie/ References

Caliò I., Gladwell G.M.L., Morassi A. (2011). Families of beams with the same buckling spectrum. *Inverse Problems*, 27, Paper 045006 (16pp).

Darboux G. (1882). Sur la répresentation sphérique des surfaces. *Comptes Rendus de l'Académie des Sciences*, 94: 1343-1345.

Love A.E.H. (1944). *A Treatise on the Mathematical Theory of Elasticity*. New York: Dover Publications Edition.

Morassi A. (2015). Exact construction of beams with a finite number of given natural frequencies. *Journal of Vibration and Control*, 21: 591-600.

Pöschel J., Trubowitz E. (1987). *Inverse Spectral Theory*. London: Academic Press.