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A rigorous justification of design formulas for torsion in thin profiles

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Abstract. A rather straightforward derivation of the Γ-limit of the torsion problem on a thin rectangle as the thickness goes to zero is obtained. The limit stresses are evaluated and the distributional nature of one of the stress components is clarified.

Key-words. Torsion, thin-walled beams, asymptotic method, Γ-convergence.

1. Introduction. As is known, exact solutions to the torsion problem are available only for few special cases. Analytical solutions were first produced by de Saint-Venant in 1855 [4] for cross sections of simple geometry such as ellipses and rectangles. Several more or less general, or easy to apply, solution methods were introduced in the following century, we refer to the dated but interesting paper by Higgins [6] for an exhaustive review on the subject.

Even when available, however, closed form solutions are hardly implemented in current engineering practice, and approximate design formulas are usually preferred; this is especially the case for thin walled beams, slender cylinders with cross section made of thin walls, which are widely employed for their high stiffness/weight ratio.

The present work deals with the simplest possible instance: that of a thin rectangle subject to torsion. For this case, the torsional stresses obtained through an approximate formula account for one half of the

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actual twisting moment only, as already noticed by Kelvin and Tait [7]. These authors computed the limit of the series form solution for vanishing thickness and ascribed the missing half of the twisting moment to tractions vanishing everywhere but at a distance “infinitely little” from the short sides of the rectangle.

It is possible to asses the validity of an approximate formula without passing through the analytical solution for finite thickness. To our knowledge, the first to do so were Rodriguez and Viaño [13, 14], who studied the limit behavior of the solutions of Poisson’s equation by means of functional analysis techniques. By studying the Dirichlet and Neumann problems associated with the torsion of a thin section they found the limit stress function and also the limit warping function. Their concise analysis does not discuss the limit stresses. Dell’Isola and Rosa [5], in view of technical applications, have recently used the asymptotic method to compute the first three terms of the expansion, leaving aside the discussion of the convergence properties.

Here we explore an alternative based on Γ-convergence which establishes the validity of the asymptotic results, stresses included, on a rigorous basis.

In recent years Γ-convergence has been extensively applied to a great variety of mechanical problems. To name some which are close to the one studied below, Γ-convergence has been used to provide justification [1, 8], extension and formulation [2, 11] of structural models involving dimension reduction. As far as torsion is concerned, we mention the deduction of Bredt formulas for single [9] and multi-cell cross sections [10] given by Morassi.

Briefly, we state the torsion problem on a rectangle, whose thickness scales with ε, in a variational form using Prandtl’s stress function. After rescaling the domain we compute the Γ-limit of the corresponding sequence of functionals as $\varepsilon \to 0$ and derive in a consistent manner not only the limit problem to be analyzed but also the proof that the sequence of minimizers of the functionals converges to the minimizer of the limit problem. Through the Γ-convergence approach we obtain: (1) a rather straightforward derivation of the limit functional, (2) the direct computation of limit stresses, (3) the clarification of the functional spaces in which the limit tractions converge, thus specifying their distributional nature. We believe that the last two items are the main results of our short paper.

2. Torsion in thin rectangular domains. We consider a Saint-Venant's torsion for a beam with rectangular cross section under torsional moment $\tilde{M}^\varepsilon$. The beam is assumed to be homogeneous, isotropic, linearly elastic with shear modulus μ and cross section $\tilde{\Omega}_\varepsilon = (-a/2, a/2) \times (-\varepsilon b/2, \varepsilon b/2)$. We are interested in characterizing the asymptotic limit of the stress distribution when the parameter ε goes to zero.

We denote with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ the base vectors of an orthogonal Cartesian system with origin in the centre of the rectangle with axes $\tilde{x}_1$ and $\tilde{x}_2$ as shown in Figure 1 We follow the formulation first proposed

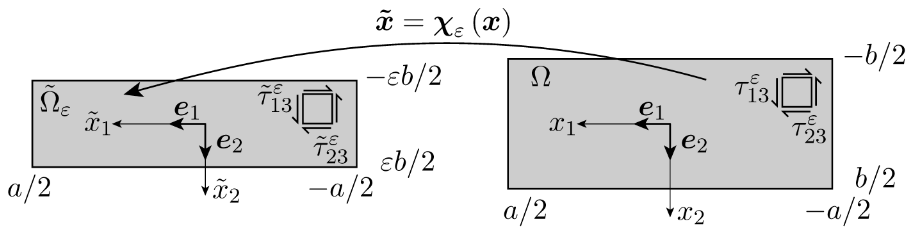

Figure 1. Actual and reference domains.

by Prandtlin1903 [12], in which the tangential traction vector $\tilde{\boldsymbol{\tau}}^\varepsilon$ is given in terms of the stress function $\tilde{\psi}^\varepsilon_m$ by

$$\tilde{\boldsymbol{\tau}}^\varepsilon = -\mu\,\alpha\left(\boldsymbol{e}_3 \times \nabla\tilde{\psi}^\varepsilon_m\right), \tag{1}$$

where α is the twist angle per unit length and $\times$ denotes the vector product.
The stress function $\tilde{\psi}^\varepsilon_m$ is determined by the boundary value problem

$$\begin{cases} \triangle\tilde{\psi}^\varepsilon_m &= -2 \quad \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{\psi}^\varepsilon_m &= 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon. \end{cases} \tag{2}$$

Let the functional $\tilde{\mathcal{F}}^\varepsilon : H^1_0(\tilde{\Omega}_\varepsilon) \to \mathbb{R}$ be defined by

$$\tilde{\mathcal{F}}^\varepsilon(\tilde{\psi}) := \int_{\tilde{\Omega}_\varepsilon} |\nabla\tilde{\psi}|^2 - 4\tilde{\psi}\, d\tilde{a}.$$

Then problem (2) can be restated in variational form as

$$\tilde{\mathcal{F}}^\varepsilon(\tilde{\psi}_m^\varepsilon) = \min_{\tilde{\psi}\in H_0^1(\tilde{\Omega}_\varepsilon)} \tilde{\mathcal{F}}^\varepsilon(\tilde{\psi}). \tag{3}$$

In order to consider the limit solution of variational problem (3) for ε tending to zero it is convenient to represent the functions on a fixed domain $\Omega = (-a/2, a/2) \times (-b/2, b/2)$ by means of the transformation of coordinates $\boldsymbol{\chi}_\varepsilon : \Omega \to \tilde{\Omega}_\varepsilon$, defined by

$$(\tilde{x}_1, \tilde{x}_2) = \boldsymbol{\chi}_\varepsilon(x_1, x_2) := (x_1, \varepsilon x_2). \tag{4}$$

Map $\boldsymbol{\chi}_\varepsilon$ establishes a natural correspondence between the functions defined in the two domains $\tilde{\Omega}_\varepsilon$ and Ω

$$\tilde{\psi} = \psi \circ \boldsymbol{\chi}_\varepsilon^{-1}, \tag{5}$$

which is an isomorphism between $H_0^1(\tilde{\Omega}_\varepsilon)$ and $H_0^1(\Omega)$. Hereafter we denote by an overset tilde functions defined in the physical domain $\tilde{\Omega}_\varepsilon$. From (5) it follows that

$$\frac{\partial\tilde{\psi}}{\partial\tilde{x}_1} = \frac{\partial\psi}{\partial x_1} \circ \boldsymbol{\chi}_\varepsilon^{-1} \qquad \text{and} \qquad \frac{\partial\tilde{\psi}}{\partial\tilde{x}_2} = \frac{1}{\varepsilon}\frac{\partial\psi}{\partial x_2} \circ \boldsymbol{\chi}_\varepsilon^{-1} \tag{6}$$

By changing variables and taking (6) into account, the functional $\tilde{\mathcal{F}}^\varepsilon$ becomes

$$\tilde{\mathcal{F}}^\varepsilon(\tilde{\psi}) = \varepsilon \int_\Omega \left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{1}{\varepsilon}\frac{\partial\psi}{\partial x_2}\right)^2 - 4\psi \, da =: \mathcal{F}^\varepsilon(\psi), \tag{7}$$

which defines a new functional $\mathcal{F}^\varepsilon : H_0^1(\Omega) \to \mathbb{R}$. The sought stress function $\psi_m^\varepsilon = \tilde{\psi}_m^\varepsilon \circ \boldsymbol{\chi}_\varepsilon^{-1}$ now minimizes the functional $\mathcal{F}^\varepsilon$, that is,

$$\mathcal{F}^\varepsilon(\psi_m^\varepsilon) = \min_{\psi\in H_0^1(\Omega)} \mathcal{F}^\varepsilon(\psi). \tag{8}$$

3. The limit problem. We shall now study the limit behaviour of the variational problem defined in equations (7) and (8). We consider a sequence of potentials ψ^{ε}[§] such that functional $\mathcal{F}^{\varepsilon}/\varepsilon^3$ is bounded in the space

$$W \;:=\; L^2\left(\left(-\tfrac{a}{2},\tfrac{a}{2}\right); H_0^1\left(-\tfrac{b}{2},\tfrac{b}{2}\right)\right),$$

endowed with norm

$$\|\psi\|_W^2 = \int_{-a/2}^{a/2} \|\psi\|^2_{L^2(-b/2,b/2)} + \left\|\frac{\partial\psi}{\partial x_2}\right\|^2_{L^2(-b/2,b/2)} \; dx_1 .$$

Let us first state an auxiliary result.

Inequality 1 (Poincaré-like)**.**
For all g in $H_0^1(\Omega)$ $: \|g\|_{L^2(\Omega)} \le b \left\|\frac{\partial g}{\partial x_2}\right\|_{L^2(\Omega)}$.

Proof. Let at first g belong to $H_0^1(\Omega) \cap C^\infty(\Omega)$, then

$$g(x_1,x_2) - \cancel{g(x_1,-b/2)} = \int_{-b/2}^{x_2} \frac{\partial g}{\partial x_2}(x_1,s)\; ds\,.$$

By Jensen's inequality,

$$g^2 \le \left(b \fint_{-b/2}^{b/2} \left|\frac{\partial g}{\partial x_2}(x_1,s)\right| ds \right)^2 \le b \int_{-b/2}^{b/2} \left(\frac{\partial g}{\partial x_2}(x_1,s)\right)^2 ds\,.$$

Integrating over the domain Ω both the left and right hand side of the above inequality one obtains:

$$\|g\|^2_{L^2} \le b^2 \left\|\frac{\partial g}{\partial x_2}\right\|^2_{L^2}$$

By density the result.

Lemma 2 (Boundedness)**.**
Let $\{\psi^\varepsilon\} \subset H_0^1(\Omega)$ be a sequence such that $\sup_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} < +\infty$. *Then*

$$\sup_\varepsilon \left\| \frac{1}{\varepsilon}\frac{\partial \psi^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)} < +\infty \qquad \textit{and} \qquad \sup_\varepsilon \left\|\frac{\psi^\varepsilon}{\varepsilon^2}\right\|_W < +\infty\,.$$

[§] With a slight abuse of notation we use to call sequences families indicized by a continuous parameter $\varepsilon \in (0,1)$.

Proof. By assumption and by (7)

$$+\infty > \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} = \int_\Omega \left(\frac{1}{\varepsilon}\frac{\partial\psi^\varepsilon}{\partial x_1}\right)^2 + \left(\frac{1}{\varepsilon^2}\frac{\partial\psi^\varepsilon}{\partial x_2}\right)^2 - 4\,\frac{\psi^\varepsilon}{\varepsilon^2}\,da$$

By using Young's inequality: $c\,d \le \delta\,c^2 + \frac{1}{4\,\delta}\,d^2$ for all $\delta > 0$, and Inequality 1, we get

$$\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \ge \int_\Omega \left(\frac{1}{\varepsilon}\frac{\partial\psi^\varepsilon}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{1}{\varepsilon^2}\frac{\partial\psi^\varepsilon}{\partial x_2}\right)^2 + \frac{1}{2\,b^2}\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2 - \delta\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2 - \frac{1}{4\,\delta}\,da.$$

By choosing $1/\delta = 4b^2$ we obtain a sum of squared terms on the right hand side

$$+\infty > b^2 + \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} \ge \int_\Omega \left(\frac{1}{\varepsilon}\frac{\partial\psi^\varepsilon}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{1}{\varepsilon^2}\frac{\partial\psi^\varepsilon}{\partial x_2}\right)^2 + \frac{1}{4\,b^2}\left(\frac{\psi^\varepsilon}{\varepsilon^2}\right)^2\,da$$

which implies the thesis.

As a consequence of Lemma 2 we notice that the derivatives of stress potential ψ^ε with respect to x_1 and x_2 appear to be rescaled by two different powers of ε, 1 and 2 respectively. This fact has relevant consequences on forthcoming results.
Owing to the weak compactness of L^2 and W it follows

Lemma 3 (Compactness)**.**
For any sequence $\{\psi^\varepsilon\} \subset H_0^1(\Omega)$ *satisfying* $\sup_\varepsilon \frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3} < +\infty$, *there exist a* $\psi \in W$ *and a subsequence of* $\{\psi^\varepsilon\}$, *not relabeled, such that*

$$\frac{\psi^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi \quad \text{and} \quad \frac{1}{\varepsilon}\frac{\partial\psi^\varepsilon}{\partial x_1} \xrightarrow{L^2(\Omega)} 0.$$

Proof. From Lemma 2 we deduce the existence of a $\psi \in W$ and a $\xi \in L^2(\Omega)$ such that

$$\frac{\psi^\varepsilon}{\varepsilon^2} \xrightarrow{W} \psi \quad \text{and} \quad \frac{\frac{\partial\psi^\varepsilon}{\partial x_1}}{\varepsilon} \xrightarrow{L^2(\Omega)} \xi.$$

But

$$\int_\Omega \xi\,\eta = \lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\varepsilon}\frac{\partial\psi^\varepsilon}{\partial x_1}\,\eta = -\lim_{\varepsilon\to 0}\,\varepsilon\int_\Omega \frac{\psi^\varepsilon}{\varepsilon^2}\frac{\partial\eta}{\partial x_1} = 0 \qquad \forall\eta \in \mathcal{C}_0^\infty(\Omega),$$

and thus $\xi = 0$.

By Lemma 2 the functional $\frac{\mathcal{F}^\varepsilon}{\varepsilon^3}$, thought of as a functional of $\psi^\varepsilon/\varepsilon^2$, is equicoercive with respect to the weak topology in W and by Proposition 8.10 of Dal Maso [3] we can characterize the Γ-limit in terms of weakly converging sequences. Thus $\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3}$ Γ-converges to the functional $\mathcal{F}^0 : W \to \mathbb{R}$ in the weak topology of W if $\mathcal{F}''(\psi) \le \mathcal{F}^0(\psi) \le \mathcal{F}'(\psi)$ for any $\psi \in W$, where

$$\begin{aligned}
\mathcal{F}'(\psi) &:= \Gamma - \liminf_{\varepsilon\to 0} \frac{\mathcal{F}^\varepsilon}{\varepsilon^3}(\psi) \\
&:= \inf\Big\{ \liminf_{\varepsilon_j\to 0} \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi^{\varepsilon_j}) : \text{ par } \frac{\psi^{\varepsilon_j}}{\varepsilon_j^2} \rightharpoonup \psi \text{ in } W\Big\}, \\
\mathcal{F}''(\psi) &:= \Gamma - \limsup_{\varepsilon\to 0} \frac{\mathcal{F}^\varepsilon}{\varepsilon^3}(\psi) \\
&:= \inf\Big\{ \limsup_{\varepsilon_j\to 0} \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi^{\varepsilon_j}) : \text{ par } \frac{\psi^{\varepsilon_j}}{\varepsilon_j^2} \rightharpoonup \psi \text{ in } W\Big\}.
\end{aligned}$$

Theorem 4 (Γ– convergence)**.** *Let $\mathcal{F}^0 : W \to \mathbb{R}$ be defined by*

$$\mathcal{F}^0(\psi) := \int_\Omega \left(\frac{\partial\psi}{\partial x_2}\right)^2 - 4\psi\, da.$$

Then $\frac{\mathcal{F}^\varepsilon(\psi^\varepsilon)}{\varepsilon^3}$ Γ-converges to $\mathcal{F}^0$ in the weak topology of W.

Proof. We start by proving that $\mathcal{F}^0(\psi) \le \mathcal{F}'(\psi)$ for any $\psi \in W$. Let $\psi \in W$ and $\psi^{\varepsilon_j} \rightharpoonup \psi$ in W. Then by the weak lower semicontinuity of the norm of W we have

$$\liminf_{\varepsilon_j\longrightarrow 0} \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi^{\varepsilon_j}) \ge \liminf_{\varepsilon_j\longrightarrow 0} \int_\Omega \left(\frac{1}{\varepsilon_j^2}\frac{\partial\psi^{\varepsilon_j}}{\partial x_2}\right)^2 - 4\,\frac{\psi^{\varepsilon_j}}{\varepsilon_j^2}\, da \ge \mathcal{F}^0(\psi).$$

To prove $\mathcal{F}''(\psi) \le \mathcal{F}^0(\psi)$ let us first assume that $\psi \in \mathcal{C}_0^\infty(\Omega)$. Consider the sequence $\psi^{\varepsilon_j} = \varepsilon_j^2\,\psi$. Then

$$\limsup_{\varepsilon_j\longrightarrow 0} \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi^{\varepsilon_j}) = \limsup_{\varepsilon_j\longrightarrow 0} \int_\Omega \varepsilon_j^2 \frac{\partial\psi}{\partial x_1}^2 + \frac{\partial\psi}{\partial x_2}^2 - 4\,\psi = \mathcal{F}^0(\psi)$$

If $\psi \in W \backslash \mathcal{C}_0^\infty(\Omega)$ there is a sequence $\{\psi_k\} \subset \mathcal{C}_0^\infty(\Omega)$ converging strongly in W to ψ. Since, by the equation above, $\mathcal{F}''(\psi_k) \le \mathcal{F}^0(\psi_k)$, the weak lower semicontinuity of $\mathcal{F}''$ and the continuity of $\mathcal{F}^0$ respect to the strong convergence in W implies

$$\mathcal{F}''(\psi) \leq \liminf_{k\to+\infty} \mathcal{F}''(\psi_k) \leq \liminf_{k\to+\infty} \mathcal{F}^0(\psi_k) = \mathcal{F}^0(\psi)$$

and the proof is concluded.

4. Convergence of the minimizers. Let $\varepsilon_j \to 0$. Since $\frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi_m^{\varepsilon_j}) \leq \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(0) = 0$, by Lemma 2 the sequence $\{\psi_m^{\varepsilon_j}/\varepsilon_j^2\}$ is equibounded in W. Thus there exists a subsequence weakly converging in W to a function ψ_m. By Γ-convergence, see Dal Maso [3, Corollary 7.17], ψ_m is the minimizer of $\mathcal{F}^0$ and

$$\lim_{\varepsilon_j\longrightarrow 0} \frac{\mathcal{F}^{\varepsilon_j}}{\varepsilon_j^3}(\psi_m^{\varepsilon_j}) = \mathcal{F}^0(\psi_m). \tag{9}$$

Since the limit function ψ_m does not depend on the chosen subsequence we have that the full sequence converges. From the strict convexity of the functionals $\mathcal{F}^\varepsilon$ we deduce the strong convergence of the minimizers.

Theorem 5. *With the notation above, we have*

$$\frac{\psi_m^\varepsilon}{\varepsilon^2} \to \psi_m, \qquad \text{in } W. \tag{10}$$

Proof. With simple estimates we find,

$$\frac{\mathcal{F}^\varepsilon}{\varepsilon^3}(\psi_m^\varepsilon) \geq \int_\Omega \Big(\frac{1}{\varepsilon^2}\frac{\partial\psi_m^\varepsilon}{\partial x_2}\Big)^2 - 4\,\frac{\psi_m^\varepsilon}{\varepsilon^2}\,da = \int_\Omega \Big(\frac{1}{\varepsilon^2}\frac{\partial\psi_m^\varepsilon}{\partial x_2} - \frac{\partial\psi_m}{\partial x_2}\Big)^2 + \\ + 2\frac{\partial\psi_m}{\partial x_2}\Big(\frac{1}{\varepsilon^2}\frac{\partial\psi_m^\varepsilon}{\partial x_2} - \frac{\partial\psi_m}{\partial x_2}\Big) + \Big(\frac{\partial\psi_m}{\partial x_2}\Big)^2 - 4\,\frac{\psi_m^\varepsilon}{\varepsilon^2}\,da.$$

Taking the limit of both sides and using (9) we deduce

$$0 \geq \lim_{\varepsilon\longrightarrow 0} \int_\Omega \Big(\frac{1}{\varepsilon^2}\frac{\partial\psi_m^\varepsilon}{\partial x_2} - \frac{\partial\psi_m}{\partial x_2}\Big)^2\,da.$$

Using Poincare's inequality we conclude the proof.

Imposing the stationarity condition for $\mathcal{F}^0$, we find that ψ_m satisfies

$$\begin{cases} \dfrac{\partial^2 \psi}{\partial x_2^2} = -2, \\ \psi\left(\cdot, -\frac{b}{2}\right) = \psi\left(\cdot, \frac{b}{2}\right) = 0, \end{cases}$$

whose solution is

$$\psi_m = -\left(x_2^2 - \frac{b^2}{4}\right) \tag{11}$$

It is worth noticing that the Dirichlet boundary condition on the short sides of the rectangle ($\psi\left(\pm a/2, \cdot\right) = 0$) is not imposed in the limit problem. This is due to the lack of control of terms containing the derivative of the stress potential $\psi^\varepsilon/\varepsilon^2$ with respect to x_1 in functionals $\frac{\mathcal{F}^\varepsilon}{\varepsilon^3}$. Even if ψ is a priori a function of both x_1 and x_2, the solution ψ_m depends only on the latter, confirming the violation of boundary constraints in $x_1 = \pm a/2$.

Figure 2 illustrates three scaled solutions $\psi_m^\varepsilon/\varepsilon^2$ plotted on the reference domain Ω for decreasing values of ε. The convergence towards the minimizer ψ_m of the limit functional $\mathcal{F}^0$, see (11), is evident.

5. Limit stresses. In this section we derive the limit stresses. To this end it is first necessary to give a consistent definition of tractions τ_{13}^ε and τ_{23}^ε in the reference domain Ω, this will be achieved by relating the stresses τ_{13}^ε and τ_{23}^ε in the reference domain Ω with those in the actual domain $\tilde{\Omega}_\varepsilon$.

Component-wise definition of $\tilde{\tau}_{13}^\varepsilon$ and $\tilde{\tau}_{23}^\varepsilon$ is, recalling (1), given by

$$\tilde{\tau}_{13}^\varepsilon = \mu\alpha \frac{\partial \tilde{\psi}_m^\varepsilon}{\partial \tilde{x}_2}, \qquad \tilde{\tau}_{23}^\varepsilon = -\mu\alpha \frac{\partial \tilde{\psi}_m^\varepsilon}{\partial \tilde{x}_1}.$$

By making the change of variables (4) and (5) and taking into account (10) it is natural to define

$$\tau_{13}^\varepsilon := \frac{\tilde{\tau}_{13}^\varepsilon}{\varepsilon} \circ \boldsymbol{\chi}_\varepsilon = \frac{\mu\alpha}{\varepsilon^2} \frac{\partial \psi_m^\varepsilon}{\partial x_2}. \tag{12}$$

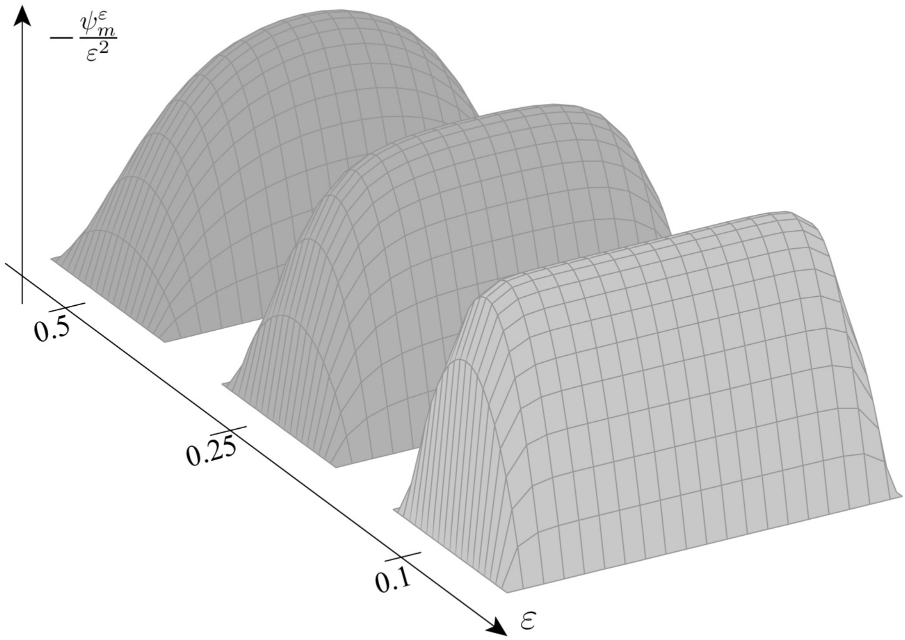

Figure 2. Convergence of the minimizers.

To determine the right scaling of τ_{23}^ε we look at the equilibrium equation, written in a weak form, and we perform a change of variables to find

$$\int_{\tilde{\Omega}_\varepsilon} \tilde{\tau}_{13}^\varepsilon \frac{\partial \tilde{\eta}}{\partial \tilde{x}_1} + \tilde{\tau}_{23}^\varepsilon \frac{\partial \tilde{\eta}}{\partial \tilde{x}_2} d\tilde{a} = \int_\Omega (\tilde{\tau}_{13}^\varepsilon \circ \boldsymbol{\chi}_\varepsilon \frac{\partial \eta}{\partial x_1} + \frac{\tilde{\tau}_{23}^\varepsilon}{\varepsilon} \circ \boldsymbol{\chi}_\varepsilon \frac{\partial \eta}{\partial x_2}) \varepsilon \, da = 0, \quad \forall \eta \in H^1(\Omega)$$

where the relation between $\tilde{\eta}$ and η is given by (5). Thus, taking into account (12), we define

$$\tau_{23}^\varepsilon := \frac{\tilde{\tau}_{23}^\varepsilon}{\varepsilon^2} \circ \boldsymbol{\chi}_\varepsilon = -\frac{\mu\alpha}{\varepsilon^2} \frac{\partial \psi_m^\varepsilon}{\partial x_1}$$

As first occurred in lemma 2 we observe that stress components rescale with different powers of ε in passing from the actual to the reference domain. This is expected since only one side of the rectangular domain shrinks to zero.

We now study the limits of the stresses τ_{13}^ε and τ_{23}^ε.

The limit of τ_{13}^{ε} follows at once from (10). Indeed we have

$$\tau_{13}^{\varepsilon} = \mu\alpha \frac{1}{\varepsilon^2} \frac{\partial \psi_m^{\varepsilon}}{\partial x_2} \to \mu\alpha \frac{\partial \psi_m}{\partial x_2} =: \tau_{13} \quad \text{in } W.$$

Expression of the potential ψ_m given in (11) yields

$$\tau_{13} = -2\mu\alpha\, x_2, \quad \text{in } \Omega. \tag{13}$$

The case of τ_{23}^{ε} is slightly more involved because the existence of a limit is not assured by the Γ-convergence theorem. Let

$$H^* := H^1((-\frac{a}{2}, \frac{a}{2}); L^2(-\frac{b}{2}, \frac{b}{2}))^*$$

denote the dual space of

$$H := H^1((-\frac{a}{2}, \frac{a}{2}); L^2(-\frac{b}{2}, \frac{b}{2})).$$

Then

$$\left\| \frac{1}{\varepsilon^2} \frac{\partial \psi_m^{\varepsilon}}{\partial x_1} \right\|_{H^*} = \sup_{\eta \in H} \frac{\int_{\Omega} \psi_m^{\varepsilon} \frac{\partial \eta}{\partial x_1} \frac{1}{\varepsilon^2}\, da}{\|\eta\|_H} \leq \left\| \frac{\psi_m^{\varepsilon}}{\varepsilon^2} \right\|_{L^2(\Omega)} < +\infty$$

Hence from the bound above it follows that there exists a subsequence of $\frac{1}{\varepsilon^2} \frac{\partial \psi_m^{\varepsilon}}{\partial x_1}$ weakly convergent in H^*. Indeed using the fact that $\psi_m^{\varepsilon}/\varepsilon^2$ is converging in $L^2(\Omega)$, we deduce that $\frac{1}{\varepsilon^2} \frac{\partial \psi_m^{\varepsilon}}{\partial x_1}$ is a Cauchy sequence in H^* and thus it converges strongly.

Let us denote by $\mathcal{H}^1$ the one dimensional Hausdorff measure, and by

$$B_a^+ := \left\{ \tfrac{a}{2} \right\} \times \left(-\tfrac{b}{2}, \tfrac{b}{2} \right), \qquad B_a^- := \left\{ -\tfrac{a}{2} \right\} \times \left(-\tfrac{b}{2}, \tfrac{b}{2} \right),$$

the end sides of the rectangle in direction x_1. We claim that

$$\tau_{23}^{\varepsilon} \to \tau_{23} \quad \text{in } H^*, \tag{14}$$

where

$$\tau_{23} := \mu\, \alpha\, \psi_m \left(\mathcal{H}^1 \llcorner B_a^+ - \mathcal{H}^1 \llcorner B_a^- \right), \tag{15}$$

that is the element of H^* defined by

$$\langle \tau_{23}, \eta \rangle = \mu\alpha \int_{-b/2}^{b/2} \psi_m \left[\eta\left(\tfrac{a}{2}, x_2 \right) - \eta\left(-\tfrac{a}{2}, x_2 \right) \right] dx_2, \quad \forall \eta \in H.$$

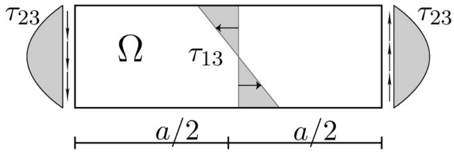

Figure 3. Limit traction distribution.

Indeed we have

$$\mu\alpha \sup_{\eta\in H} \frac{\int_\Omega \psi_m^\varepsilon \frac{\partial\eta}{\partial x_1} \frac{1}{\varepsilon^2}\, da - \int_{-b/2}^{b/2} \psi_m \left[\eta\left(\frac{a}{2}, x_2\right) - \eta\left(-\frac{a}{2}, x_2\right)\right] dx_2}{\|\eta\|_H} =$$

$$= \mu\alpha \sup_{\eta\in H} \frac{\int_\Omega \left(\frac{\psi_m^\varepsilon}{\varepsilon^2} - \psi_m\right) \frac{\partial\eta}{\partial x_1}\, da}{\|\eta\|_H} = \|\tau_{23}^\varepsilon - \tau_{23}\|_{H^*} \leq \left\| \frac{\psi_m^\varepsilon}{\varepsilon^2} - \psi_m \right\|_{L^2(\Omega)},$$

from which follows claim (14).

The limit stress field resulting from (13) and (15) is schematically illustrated in Figure 3 Stresses τ_{13} are linearly distributed in the thickness and uniform along the rectangle's mean line $x_2 = 0$, while τ_{23} is a measure supported at sides $x_1 = \pm a/2$ where it is distributed parabolically.

6. Twisting moment and torsional stiffness. The applied torsional moment $\tilde{M}^\varepsilon$ can be computed by either of the two following equations

$$\tilde{M}^\varepsilon = \int_{\tilde{\Omega}_\varepsilon} \tilde{x}_1\, \tilde{\tau}_{23}^\varepsilon - \tilde{x}_2\, \tilde{\tau}_{13}^\varepsilon\, d\tilde{a} = 2\,\mu\,\alpha \int_{\tilde{\Omega}_\varepsilon} \tilde{\psi}_m^\varepsilon\, da.$$

Let $M^\varepsilon := \frac{\tilde{M}^\varepsilon}{\varepsilon^3}$ be the rescaled moment, which can be written as

$$M^\varepsilon := \frac{\tilde{M}^\varepsilon}{\varepsilon^3} = \int_\Omega x_1\, \tau_{23}^\varepsilon - x_2\, \tau_{13}^\varepsilon\, da = 2\,\mu\,\alpha \int_\Omega \frac{\psi_m^\varepsilon}{\varepsilon^2}\, da.$$

By means of the convergences established in the previous sections we find that M^ε converges to a moment M which can be written as

$$M = 2\,\mu\,\alpha \int_\Omega \psi_m\, da = \mu\,\alpha\, \frac{a\, b^3}{3},$$

or as

$$M = \langle \tau_{23}, x_1 \rangle - \int_\Omega x_2\, \tau_{13}\, da,$$

where the two contributions on the right hand side are given by

$$\langle \tau_{23}, x_1 \rangle = \mu\, \alpha \int_{-b/2}^{b/2} \psi_m\, dx_2 \left(\frac{a}{2} - \left(-\frac{a}{2} \right) \right) = \mu\, \alpha\, \frac{a\, b^3}{6}$$

$$- \int_\Omega x_2\, \tau_{13}\, da = 2\, \mu\, \alpha \int_\Omega x_2^2 da = \mu\, \alpha\, \frac{a\, b^3}{6}.$$

The two stress components are found to each account for half of the overall stiffness; in particular it emerges that the τ_{23} contribution is that of a couple of forces $F = \mu\, \alpha \int_{-b/2}^{b/2} \psi_m\, dx_2$ acting on the short sides of the domain.

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